Brauer Groups, Root Stacks, and the Chow Ring of the Stack of Expanded Pairs

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Abstract

Algebraic stacks are a type of object studied in algebraic geometry which encode certain geometric moduli problems, in the sense that they keep track of both families of objects and automorphisms of those families. This thesis, consisting of an introduction and three parts, is a collection of works concerning algebraic stacks. The results presented here come in two flavors: firstly, using algebraic stacks as a tool to develop solutions for classical problems, secondly, studying the geometry of algebraic stacks in their own right.

The first two parts concern the Brauer group, a classical invariant for fields, which has a generalization to schemes and Deligne-Mumford stacks using étale cohomology. In the first part, we use root stacks, destackification, and resolution of singularities to generalise an existing result about standard forms of conic bundles to the case of a not necessarily algebraically closed base field.

In the second part, we give an interpretation of the residue map from the Brauer group of the quotient field of a discrete valuation ring to the cohomology of the residue field in terms of root stacks and Weil restriction. The given description of the residue map has a geometric application to the residue map for Severi-Brauer bundles in standard form.

In the third part, we study a particular example of a moduli stack, namely the stack of expanded pairs $\mathcal{T}$, which was first studied in the context of degeneration formulas in Gromov-Witten theory. We use a logarithmic description of $\mathcal{T}$ to calculate its Chow ring, which we find to be the ring of quasisymmetric functions QSym originating in combinatorics. We construct a monoid structure on $\mathcal{T}$, which is shown to induce the comultiplication for the Hopf algebra structure on QSym. Finally, we use an interpretation of $\mathcal{T}$ in terms of moduli stacks of curves to obtain the Chow rings of certain moduli stacks of semistable curves.

Abstract

Algebraische Stacks sind eine Art von Objekt, welches in der algebraischen Geometrie studiert wird. Stacks stellen die Lösung für gewisse Modulprobleme dar, indem sie sowohl Familien von Objekten als auch deren Automorphismen enkodieren. Diese Arbeit, bestehend aus einer Einführung und drei Teilen, ist eine Sammlung von verschiedenen Arbeiten über algebraische Stacks. Es werden zwei verschiedene Typen von Resultaten präsentiert: erstens der Gebrauch von Stacks als Werkzeug, um Probleme in klassischer algebraischer Geometrie zu lösen, zweitens das Studium der Geometrie von Stacks als Objekt von separatem Interesse.

In den ersten zwei Teilen geht es um die Brauergruppe, eine klassische Invariante für Körper. Diese besitzt eine Verallgemeinerung auf Schemas und Deligne-Mumford-Stacks mithilfe von Étale Kohomologie. Im ersten Teil benutzen wir Wurzelstacks, Destackifizierung, und Auflösung von Singularitäten, um ein klassiches Resultat über Standardformen von konischen Bündeln für den Fall eines nicht unbedingt algebraisch abgeschlossenen Grundkörpers zu verallgemeinern.

Im zweiten Teil geben wir eine Interpretation der Residuenabbildung von der Brauergruppe des Quotientenkörpers eines diskreten Valuationsrings in die Kohomologie des Restklassenkörpers, mithilfe von Wurzelstacks und Weil-Einschränkung. Die erhaltene Beschreibung der Residuenabbildung hat eine geometrische Anwendung für die Residuenabbildung von Severi-Brauer-Bündeln in Standardform.

Im dritten Teil untersuchen wir ein konkretes Beispiel eines Modulstacks, nämlich den Stack von erweiterten Paaren $\mathcal{T}$, der zuerst im Zusammenhang mit Degenerationsformeln in der Gromov-Witten-Theorie untersucht wurde. Wir benutzen eine logarithmische Beschreibung von $\mathcal{T}$, um dessen Chow-Ring zu berechnen, den wir als den Ring der quasisymmetrischen Funktionen QSym identifizieren, welcher seinen Ursprung in der Kombinatorik hat. Wir konstruieren eine Monoidstruktur auf $\mathcal{T}$ und zeigen, dass diese die Komultiplikation für die Hopf-Algebra-Struktur auf QSym induziert. Zuletzt benutzen wir eine Interpretation von $\mathcal{T}$ als Modulstack von Kurven, um die Chow-Ringe von gewissen Modulstacks von semistabilen Kurven zu erhalten.

To Robin Morgaine

"It is well known that a vital ingredient of success is not knowing that what you are attempting cannot be done." – Terry Pratchett

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Introduction

Since the introduction of what is today called a Deligne-Mumford stack in the groundbreaking paper by Deligne and Mumford ([8]) and the generalization to an algebraic stack by Artin ([4]), algebraic stacks have become an important tool in algebraic geometry.

In this thesis, we exhibit two types of results connected to the theory of algebraic stacks. The first one (treated in papers A and B) concerns the application of algebraic stacks to classical problems in algebraic geometry. More precisely, we use the connection between the theory of Brauer groups and the root stack construction to reinterpret questions about Brauer classes defined on an open subset of certain varieties as questions about Brauer classes on the root stack of the variety along the complement of the open subset. This allows us to generalize a result by Sarkisov ([27]) about standard forms of conic bundles (with no reference to algebraic stacks in the statement), and to formulate the Brauer residue map in terms of the root stack.

The second type (treated in paper C) concerns the investigation of the geometric nature of a specific algebraic stack: we consider the stack of expansions $\mathcal{T}$, a variant of which was first studied by Li and Ruan ([22]) in symplectic geometry in the context of Gromov-Witten theory, which has later been proven to be an algebraic stack. We use another description related to logarithmic geometry to compute its Chow ring, which we find to be the ring of quasi-symmetric functions QSym, originally defined by Gessel ([9]). The ring QSym, which has been well-studied from the perspective of combinatorics, has the structure of a graded Hopf algebra. We detail how this Hopf algebra structure arises geometrically on $\mathcal{T}$.

In what follows, we will give a more detailed introduction to subjects treated in these papers.

Conventions and notation

We assume knowledge of basic notions in algebraic geometry, as for example in [15]. A detailed introduction to the theory of algebraic stacks, including the root stack construction used in this work, can be found in [26]. For our purposes, an *algebraic stack* $\mathcal{X}$ is a stack in groupoids in the étale topology such that the diagonal $\Delta_{\mathcal{X}}$ is representable and such that there exists a smooth surjection from a scheme to $\mathcal{X}$;

we do not impose any other conditions on the diagonal. A *Deligne-Mumford stack* is an algebraic stack that admits an *étale* surjection from a scheme.

The Brauer group

A classical reference for the theory of Brauer groups is [25], see also the series of papers [12, 13, 14]. Let k be any field. A *central simple algebra* over k is a finite-dimensional associative algebra over k which has no non-trivial two-sided ideals and whose center is equal to k. It is a consequence of the Artin-Wedderburn theorem that every central simple algebra over k is isomorphic to a matrix algebra $M_n(S)$ over a division ring S over k. Defining an equivalence relation on central simple algebras by declaring that $A \sim B$ if $M_{n_1}(A) \cong M_{n_2}(B)$, a relation called *similarity*, we obtain the *Brauer group* $\mathrm{Br}(k)$, with group structure given by the tensor product of algebras. For example, the Brauer group of an algebraically closed field $\bar{k}$ is trivial, since there are no nontrivial division algebras over $\bar{k}$. The Brauer group $\mathrm{Br}(\mathbb{R})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, with a representative for the nontrivial element given by the quaternions. In fact, the Brauer group is always a torsion group.

This classical invariant can also be defined in terms of Galois cohomology: the group described above is isomorphic to

$$H^2(\mathrm{Gal}(k_{\mathrm{sep}}/k), k_{\mathrm{sep}}^*) = H^2_{et}(\mathrm{Spec}\, k, \mathbb{G}_m)$$

by associating to a central simple algebra of dimension n^2 the image of its class in $H^1_{et}(\mathrm{Spec}\, k, PGL(n))$ under the boundary homomorphism to $H^2_{et}(\mathrm{Spec}\, k, \mathbb{G}_m)$.

Both definitions generalize to a (locally Noetherian) scheme X in different ways. A sheaf A of associative $\mathcal{O}_X$-algebras is called an *Azumaya algebra* if it is locally free of finite rank as an $\mathcal{O}_X$-module and if A_x is a central simple algebra over the residue field of x for every point $x \in X$. This leads to the *(Azumaya) Brauer group* $\mathrm{Br}(X)$. On the other hand, the torsion subgroup $H^2_{et}(X, \mathbb{G}_m)$ (this is always a torsion group if X is regular Noetherian) is called the *cohomological Brauer group* of X.

Both notions evidently generalize the definition for the spectrum of a field, and they are known to agree in many geometric cases.[1] The cohomological definition generalizes to a Noetherian Deligne-Mumford stack $\mathcal{X}$, taking cohomology on the étale site of $\mathcal{X}$ as our cohomology theory. Using standard arguments, it is straightforward to see that $H^2(\mathcal{X}, \mathbb{G}_m)$ is a torsion group if $\mathcal{X}$ is regular and has trivial generic stabilizer (using [13], cf. section 2.2 of Paper A), and in general we define the Brauer group of a Deligne-Mumford stack $\mathcal{X}$ to be the torsion subgroup of $H^2(\mathcal{X}, \mathbb{G}_m)$. Much of the theory of Brauer groups of schemes can be transferred over to Deligne-Mumford stacks.

[1] In general, there will be an injection from the Azumaya Brauer group into the cohomological Brauer group, which is an isomorphism e.g. for smooth quasi-projective varieties.

Severi-Brauer varieties and bundles

A *Severi-Brauer variety*, also called Brauer-Severi variety, over a field k is a projective variety over k which is a form of projective space over k, i.e. it becomes isomorphic to $\mathbb{P}^n$ upon base change to the algebraic closure of k. A simple example is the $\mathbb{R}$-conic

$$V(x_0^2 + x_1^2 + x_2^2) \subset \mathbb{P}^2_{\mathbb{R}},$$

which has no $\mathbb{R}$-points, but becomes isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ upon base change to $\mathbb{C}$. Its class in $H^1_{et}(\operatorname{Spec}\mathbb{R}, PGL(2))$ corresponds to the class of the quaternions. More generally, it is a classical fact that isomorphism classes of Severi-Brauer varieties over k of dimension n are in bijection with central simple algebras of degree $n+1$ over k (cf. [7, 10]), which connects the theory of Severi-Brauer varieties to Brauer groups. Similarly, there is a geometric equivalence relation between Severi-Brauer varieties which is equivalent to Brauer equivalence of the corresponding Brauer classes. Apart from the evident generalization to a Severi-Brauer scheme, where we require that all fibers be Severi-Brauer varieties, there is also the more general notion of a Brauer-Severi bundle, which is a flat bundle whose generic fiber is a Brauer-Severi variety.

The simplest case ($n = 1$ in the notation above) is then the classical case of a conic bundle. Conic bundles have been widely studied as a source of examples for rationality problems, perhaps most famously by Artin and Mumford in [5], where the authors produced a unirational, non-rational conic bundle over a rational surface. In general, given a conic bundle $\pi\colon V \to S$ over a nonsingular variety S, the fibers over the singular locus need not be irreducible, and V cannot be expected to be nonsingular. A conic bundle is called *standard* if V is nonsingular and the preimage of an irreducible divisor under π is an irreducible divisor. Sarkisov proved the following:

Theorem 1 ([27]). *Let k be an algebraically closed field of characteristic not equal to 2, and let S be a smooth projective variety over k. Assume embedded resolution of singularities over k. Given any conic bundle $\pi\colon V \to S$ over S, there is a standard conic bundle $\widetilde{\pi}\colon \widetilde{V} \to \widetilde{S}$, a birational morphism $\widetilde{S} \to S$, and a birational map $\widetilde{V} \dashrightarrow V$, making the square*

$$\begin{tikzcd} \widetilde{V} \arrow[r, dashed] \arrow[d, "\widetilde{\pi}"] & V \arrow[d, "\pi"] \\ \widetilde{S} \arrow[r] & S \end{tikzcd}$$

commute.

Our main result in Paper A is a generalization this result to the case of a field which is not necessarily algebraically closed. A similar technique has also been employed in [21] to produce an analogous result for the $n = 2$ analogue: Brauer-Severi surface bundles.

Root stacks and Brauer groups

The *root stack construction* (cf. [6, 2]) is the answer to the following problem: given a scheme X, or more generally a Deligne-Mumford stack, and an effective Cartier divisor D on X, we wish to find an n-th root of D, i.e. a divisor D' such that $nD' = D$.[2] Of course, since this is not always possible, the best we can hope for is to find a (universal) morphism $f : Y \to X$ such that D pulls back nicely to Y and such that f^*D has a n-th root.

More concretely, the n-th root stack of X along an effective Cartier divisor D, denoted by $\pi : \sqrt[n]{(X, D)} \to X$, is an algebraic stack over X which is an isomorphism over $U := X \setminus D$, and whose fiber over D is a μ_n-gerbe $D' \to D$ such that D pulls back to nD'. Locally, on an open subscheme $\mathrm{Spec}(A)$ such that $D = V(f)$, we have

$$\sqrt[n]{(\mathrm{Spec}(A), D)} = \left[\mathrm{Spec}\left(A[T]/(T^n - f)\right)/\mu_n\right].$$

If n is invertible in the local rings of X, the root stack construction produces a relative Deligne-Mumford stack. There is also a refined version, an iterated root stack for a collection $(D_1, \dots, D_m)$ of divisors, which adds more stack structure along the intersection of the divisors; it is denoted by $\sqrt[n]{(X, \{D_1, \dots, D_m\})}$ and reduces to the construction above in the case of a single divisor.

The important feature connecting the Brauer group and root stacks is the following property: under suitable regularity assumptions, any n-torsion Brauer class defined on the complement of the divisors lifts uniquely to the root stack.

This is an essential ingredient in the proof of the main theorem in Paper A. In Paper B, we use this feature to give a geometric interpretation for the Brauer residue map

$$\mathrm{res} : \mathrm{Br}(\mathrm{Spec}(K))[n] \to H^1(\mathrm{Spec}(\kappa), \mathbb{Z}/n\mathbb{Z})$$

for a discrete valuation ring with quotient field K and residue field κ.

The stack of expanded pairs

Let k be a field, and consider pairs (X, D) of a scheme X over k and a Cartier divisor D. Let $\mathcal{O}_D(D) := \mathcal{O}_X(D)|_D$, and write $P = \mathbb{P}(\mathcal{O}_D(D) \oplus \mathcal{O}_D)$ for the projectivized normal bundle of D. There are two natural sections of P at zero and infinity, with opposite normal bundles $\mathcal{O}_D(D)$ and $\mathcal{O}_D(-D)$, which we denote by D_0 and D_∞. The space P comes equipped with a $\mathbb{G}_m$-action obtained from scaling $\mathcal{O}_D(D)$.

An *expanded pair of length* $\ell \geq 0$ is the space obtained from gluing ℓ copies of P to X along D:

$$X(\ell) := X \sqcup_{D = D_\infty} \underbrace{P \sqcup_{D_0 = D_\infty} \cdots \sqcup_{D_0 = D_\infty} P}_{\ell \text{ times}},$$

[2] The general construction works in greater generality with a generalized Cartier divisor, which then also allows us to take roots of line bundles with a section.

with $X(0) = X$. A related notion is that of an expanded degeneration. These notion of families of expanded pairs and expanded degenerations originated in work of Li and Ruan in symplectic geometry ([22]) and were further studied by Li in [23, 24] in work on the degeneration formula in Gromov-Witten theory. The right definition of a family of expanded pairs in order to obtain a well-behaved moduli stack $\mathcal{T}$ is not straightforward. Graber and Vakil ([11]) defined it directly as expansions of the pair $(\mathbb{P}^1, 0)$, which gives a direct interpretation of $\mathcal{T}$ in terms of a moduli stack of semistable curves. There is also a logarithmic approach to the study of $\mathcal{T}$ ([3, 19]). The key result we use in Paper A is [1], where a good definition of the stack of expanded pairs was given and its algebraicity was proven. We use another to $\mathcal{T}$ approach related to logarithmic geometry: roughly, $\mathcal{T}$ is the moduli stack of sequences of line bundles and morphisms

$$\mathcal{L}_n \xrightarrow{s_n} \mathcal{L}_{n-1} \xrightarrow{s_{n-1}} \dots \mathcal{L}_1 \xrightarrow{s_1} \mathcal{O}$$

for $n \geq 0$, under the convention that on the locus where s_k is an isomorphism, we may identify this sequence with the sequence of length $n-1$ where $\mathcal{L}_k$ and $\mathcal{L}_{k-1}$ have been collapsed to one term.

Quasisymmetric functions

The Hopf algebra of quasisymmetric functions QSym is a well-known combinatorial graded Hopf algebra generalizing symmetric functions. A *quasisymmetric function* is an element p of the ring of formal power series in infinitely many ordered commuting variables x_i, such that for every $n > 0$, for every pair of ascending sequences $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$ of indices, and for every n-tuple of positive integers $(a_1, \dots, a_n)$, the coefficients of p for the terms $x_{i_1}^{a_1} \cdots x_{i_n}^{a_n}$ and $x_{j_1}^{a_1} \cdots x_{j_n}^{a_n}$ agree.

For example, every symmetric function is a quasisymmetric function, and

$$\sum_{i<j<k} x_i^4 x_j^3 x_k$$

is a quasisymmetric function which is not symmetric. A natural additive basis of QSym, the *monomial basis*, is parametrized by *compositions* of natural numbers. A composition of size n and weight ℓ is an ℓ-tuple of positive integers $A = (a_1, \dots, a_\ell)$ such that $a_1 + \cdots + a_\ell = n$, and it corresponds to the quasisymmetric function

$$M_A = \sum_{i_1 < \cdots < i_\ell} x_{i_1}^{a_1} \cdots x_{i_\ell}^{a_\ell}.$$

It is freely generated as an algebra over $\mathbb{Z}$ ([16, 17]) with generators in degree n in bijection with *Lyndon compositions* of size n.

It is also known that QSym is a graded Hopf algebra, whose comultiplication Δ is straightforward to describe in the monomial basis. In fact,

$$\Delta(M_A) = \sum_{A = B \cdot C} M_B \otimes M_C,$$

where $B \cdot C$ denotes concatenation of compositions. There is a well-developed combinatorial theory of QSym, and they have been used in a variety of contexts, starting with work of Gessel ([9]) on P-partitions.

Our main result relating quasisymmetric functions and expansions can be summarized as follows, using a variation on the theory of Chow groups for algebraic stacks ([20]).

Theorem 2. *The Chow ring of* $\mathfrak{T}$ *is isomorphic, as a graded ring, to the ring of quasisymmetric functions* QSym*. Moreover, there is an étale gluing morphism* $\mu : \mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$ *such that* μ^* *induces the comultiplication of* QSym *on Chow groups.*

The interpretation of $\mathfrak{T}$ as a stack of semistable genus 0 curves with three markings such that the last two markings are on the same component, due to Graber-Vakil, suggests to use this result to calculate Chow rings of stacks of genus 0 semistable marked curves, possibly extending the classical result by Keel([18]) on $\mathrm{CH}(\overline{\mathcal{M}}_{0,n})$. We calculate the Chow groups of $\mathfrak{M}^{ss}_{0,2}$ and $\mathfrak{M}^{ss}_{0,3}$ as a corollary of our result.

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Paper A: Conic bundles and Iterated Root Stacks

CONIC BUNDLES AND ITERATED ROOT STACKS

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ABSTRACT. We generalize a classical result by V. G. Sarkisov about conic bundles to the case of a not necessarily algebraically closed perfect field, using iterated root stacks, destackification, and resolution of singularities. More precisely, we prove that whenever resolution of singularities is available, over a general perfect base field, any conic bundle is birational to a standard conic bundle.

CONTENTS

1. INTRODUCTION

In this paper, we study the geometry of conic bundles, that is, fibrations whose generic fiber is a smooth conic. They have been widely studied in the context of rationality problems, notably the classic result of Artin and Mumford, who computed their Brauer groups in [5] to produce unirational non-rational conic bundles over rational surfaces. In order to better understand these bundles, it is desirable to bring a conic bundle into a standard form where the locus of degeneration can be controlled.

Over an algebraically closed field k and assuming resolution of singularities, a classical result by Sarkisov ([17]) states every conic bundle $\pi : V \to S$ of irreducible varieties such that S is smooth and π is projective can be brought into a standard form. Concretely, this means that there exist smooth varieties $\widetilde{V}$ and $\widetilde{S}$ and a projective morphism $\widetilde{\pi} : \widetilde{V} \to \widetilde{S}$ fitting

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into a commutative square

$$\begin{tikzcd} \widetilde{V} \arrow[r, dashed] \arrow[d, "\widetilde{\pi}"] & V \arrow[d, "\pi"] \\ \widetilde{S} \arrow[r] & S \end{tikzcd}$$

such that the rational map $\widetilde{V} \dashrightarrow V$ and the projective morphism $\widetilde{S} \to S$ are birational, and such that

- the generic fiber of $\widetilde{\pi}$ is a smooth conic,
- the discriminant divisor of $\widetilde{\pi}$ is a simple normal crossing divisor,
- the general fiber of $\widetilde{\pi}$ along every irreducible component of the discriminant divisor is a singular irreducible reduced conic, and
- the fibers of $\widetilde{\pi}$ over the singular locus of the discriminant divisor are non-reduced conics, i.e. double lines.

We use root stacks, resolution of singularities, and the destackification procedure ([7]) to generalize Sarkisov's result to a general perfect base field in Theorem 4. An analogous result for Brauer-Severi surface bundles, i.e. fibrations whose generic fiber is a form of $\mathbb{P}^2$, has been proven by Kresch and Tschinkel in [12], also using root stack techniques.

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2. Preliminaries

Throughout this section, when X is a Noetherian Deligne-Mumford (DM) stack, we will let n be a positive integer, invertible in the local rings of an étale atlas of X. A *sheaf* on X is a sheaf of abelian groups on the étale site of X; all cohomology will be étale cohomology.

2.1. **Conic bundles.**

Definition 1. Let S be a regular scheme such that 2 is invertible in its local rings. A *regular conic bundle over* S is a flat, projective morphism $\pi : V \to S$ from a regular scheme V such that the generic fiber over every irreducible component is smooth and such that all fibers are isomorphic to a conic in $\mathbb{P}^2$. A regular conic bundle is called *standard* if π is relatively minimal, i.e. if the preimage of an irreducible divisor under π is an irreducible divisor; equivalently, if there exists a reduced divisor $D \subset S$ whose singular locus is regular, such that

- The morphism π is smooth over $S \setminus D$ and the generic fiber over every irreducible component is a smooth conic.
- The generic fiber of π over every irreducible component of D is a singular reduced irreducible conic, i.e. the union of two lines with conjugate slopes.

- The fiber of π over every point of D^{sing} is non-reduced, i.e. a double line.

We remark that we put no further requirements on D, although our construction actually produces standard conic bundles with simple normal crossing discriminant divisor.

2.2. **Gerbes.** Let X be a Noetherian Deligne-Mumford stack.

Definition 2. A gerbe over X banded by μ_n, or simply a μ_n-gerbe over X, is the data of a Deligne-Mumford stack H and a morphism $H \to X$ that is étale locally isomorphic to a product with $B\mu_n$, together with compatible identifications of the automorphism groups of local sections with μ_n.

We can classify μ_n-gerbes by their class in $H^2(X, \mu_n)$. We use the notion of the *residual gerbe* $\mathcal{G}_x$ of X at a point $x \in |X|$, an étale gerbe over the residue field $\kappa(x)$ satisfying certain universal properties ([16, App. B]).

Lemma 1. *Let $x \in |X|$. Then $H^1(\mathcal{G}_x, \mathbb{Z}) = 0$.*

Proof. The Leray spectral sequence for $f : \mathcal{G}_x \to \operatorname{Spec} \kappa(x)$ gives a monomorphism
$$H^1(\mathcal{G}_x, \mathbb{Z}) \to H^0(\operatorname{Spec} \kappa(x), R^1 f_* \mathbb{Z}).$$
Let $K/\kappa(x)$ be a finite separable extension such that the gerbe
$$\mathcal{Y} := \operatorname{Spec} K \times_{\operatorname{Spec} \kappa(x)} \mathcal{G}_x$$
has a section. This implies that $\mathcal{Y} \cong \mathrm{B}G$ for a finite étale group scheme G over K. But then
$$H^1(\mathcal{Y}, \mathbb{Z}) = H^1(\mathrm{B}G, \mathbb{Z}) = \operatorname{Hom}(G, \mathbb{Z}) = 0.$$
Hence, we have $R^1 f_* \mathbb{Z} = 0$. □

We will also frequently make use of the *Kummer sequence*
$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{(\cdot)^n} \mathbb{G}_m \to 0,$$
which is an exact sequence of sheaves on X.

Suppose now that X is regular and integral, with trivial stabilizer at the generic point $\iota_\eta : \operatorname{Spec}(\eta) \to X$. By [10, (2)-(3)], the following is an exact sequence of sheaves on X:

$$0 \to \mathbb{G}_m \to (\iota_\eta)_* \mathbb{G}_m \to \bigoplus_{x \in X^{(1)}} (\iota_x)_* \mathbb{Z} \to 0. \tag{1}$$

By Lemma 1, this implies that $H^2(X, \mathbb{G}_m) \to H^2(X, (\iota_\eta)_* \mathbb{G}_m)$ is injective. The Leray spectral sequence for ι_η and Hilbert's Theorem 90 imply that
$$H^2(X, (\iota_\eta)_* \mathbb{G}_m) \to H^2(\operatorname{Spec}(\eta), \mathbb{G}_m)$$
is injective. Hence, we can infer that $H^2(X, \mathbb{G}_m)$ is a torsion group, which we call the *Brauer group* of X. This motivates the following definition.

Definition 3. The *Brauer group* $\mathrm{Br}(X)$ of a Noetherian DM stack X is defined to be the torsion subgroup of $H^2(X, \mathbb{G}_m)$.

This reduces to the classical definition of the Brauer group if X is the spectrum of a field.

2.3. **Root stacks and the Brauer group.** Given effective Cartier divisors

$$D_1, \ldots, D_\ell$$

on a Noetherian scheme or Deligne-Mumford stack X, we can define the *iterated n-th root stack* of X along those divisors ([8]), denoted by

$$\sqrt[n]{(X, \{D_1, \ldots, D_\ell\})} \to X.$$

This construction adds stacky structure along the divisors, and is an isomorphism outside of the union of divisors. It should be noted that when any intersection $D_i \cap D_j$ with $i \neq j$ is nonempty, this is not isomorphic to the root stack along the union of the divisors. For any $i \in \{1, \ldots, \ell\}$, the iterated root stack

$$\sqrt[n]{(X, \{D_1, \ldots, \widehat{D_i}, \ldots, D_\ell\})}$$

is a relative coarse moduli space for $\sqrt[n]{(X, \{D_1, \ldots, D_\ell\})}$ in the sense of [1, §3]. In particular, a locally free sheaf on $\sqrt[n]{(X, \{D_1, \ldots, D_\ell\})}$ such that the associated linear μ_n-representation is trivial at a general point of any irreducible component of D_i descends to

$$\sqrt[n]{(X, \{D_1, \ldots, \widehat{D_i}, \ldots, D_\ell\})}.$$

To prove this, we can apply a relative version of [2, Thm. 10.3] for good moduli space morphisms, which are relative versions of good moduli spaces. Note that

(1) the condition of trivial stabilizer actions at closed points is replaced by trivial action of the *relative* inertia stack at closed points;
(2) due to the construction of root stacks, this action will be trivial if it is trivial at a general point of every irreducible component of D_i;
(3) since the stacks involved are tame, the relative coarse moduli space mentioned above is a good moduli space morphism.

We will use this several times throughout.

Lemma 2. *Let S be a regular integral Noetherian scheme of dimension 1, let n be invertible in the local rings of S, and let $D_1, \ldots, D_\ell$ be distinct closed points of S. Then we have*

$$\mathrm{Br}(\sqrt[n]{(S, \{D_1, \ldots, D_\ell\})})[n] \cong \mathrm{Br}(S \setminus \{D_1, \ldots, D_\ell\})[n].$$

Proof. Let X be any regular integral Noetherian DM stack with trivial generic stabilizer such that $\dim(X) = 1$, and such that n is invertible in the local rings of an étale atlas of X. Then the results from [11, 2.] and the Leray spectral sequence for ι_η imply that $H^2(X,(\iota_\eta)_*\mathbb{G}_m)[n] \cong \mathrm{Br}(\eta)[n]$. Moreover, if $x \in X^{(1)}$, the vanishing of $H^1(\mathcal{G}_x,\mathbb{Z})$ implies that

$$H^1(\mathcal{G}_x,\mathbb{Z}/n\mathbb{Z}) \cong H^2(\mathcal{G}_x,\mathbb{Z})[n].$$

The long exact sequence of cohomology of (1) then gives rise to an exact sequence

$$0 \to \mathrm{Br}(X)[n] \to \mathrm{Br}(\eta)[n] \to \bigoplus_{x\in X^{(1)}} H^1(\mathcal{G}_x,\mathbb{Z}/n\mathbb{Z}). \tag{2}$$

Taking $X = \sqrt[n]{(S,\{D_1,\dots,D_\ell\})}$, with codimension 1 point x_i over D_i for all i, we compare the exact sequence (2) with the analogous exact sequence for S (loc. cit.) to obtain the vanishing of the right-hand map in (2) after projection to the factor x_i for any i [13, §3.2]. Comparison with the exact sequence for $S \setminus \{D_1,\dots,D_\ell\}$ gives the result. □

Lemma 3. *Let k be a field and let X be integral, smooth and of finite type over k with trivial generic stabilizer. For any positive integer n with $\mathrm{char}(k) \nmid n$ and open substack $U \subset X$ whose complement has codimension at least 2, we have $H^2(X,\mu_n) \cong H^2(U,\mu_n)$, and therefore $\mathrm{Br}(X)[n] \cong \mathrm{Br}(U)[n]$.*

Proof. By [14, Rem II.3.17] there is no loss of generality in assuming that k is perfect. By shrinking X if necessary and iterating the process for large open substacks of X, we can assume that the complement $Y = X \setminus U$ is smooth and of constant codimension $c \geq 2$ everywhere. In this situation, we know by [4, §XVI.3] that $\underline{H}^i_Y(X,\mu_n) = 0$ for $i \neq 2c$. Combining this with the exact sequence for cohomology with support ([14, Prop III.1.25]) and the local-to-global spectral sequence ([14, §VI.5]) gives the result; cf. [11, Cor 6.2]. □

3. Proof the of the main result

Here we state and prove the main result.

Theorem 4. *Let k be a perfect field of characteristic different from 2 and S a smooth projective algebraic variety over k. Assume that embedded resolution of singularities for reduced subschemes of S of pure codimension 1 and desingularization of reduced finite-type Deligne-Mumford stacks of pure dimension equal to $\dim(S)$ are known. Let*

$$\pi : V \to S$$

be a morphism of projective varieties over k whose generic fiber is a smooth conic. Then there exists a commutative diagram

$$\begin{tikzcd} \widetilde{V} \arrow[r, dashed, "\rho_V"] \arrow[d, "\widetilde{\pi}"] & V \arrow[d, "\pi"] \\ \widetilde{S} \arrow[r, "\rho_S"] & S \end{tikzcd}$$

where ρ_S is a projective birational morphism, ρ_V is a birational map, and $\widetilde{\pi}$ is a standard conic bundle with simple normal crossing discriminant divisor.

Remark 1. Embedded resolution of singularities is known in characteristic 0 for all dimensions by Hironaka's celebrated result. In positive characteristic, embedded resolution of singularities for both curves and surfaces is known (cf. [9]). Since the resolutions commute with smooth base change, the assumptions about desingularization of reduced finite-type Deligne-Mumford stacks are also true in all of these cases (apply resolution of singularities to a presentation).

Before embarking on the proof we make several observations of a general nature. Let k be a perfect field, let S be a smooth projective algebraic variety over k, let n be a positive integer such that $\mathrm{char}(k) \nmid n$, and let $\alpha \in \mathrm{Br}(k(S))[n]$. Then there exists a dense open $U \subset S$ and $\beta \in \mathrm{Br}(U)$ such that α is the restriction of β. Taking U to be maximal, by Lemma 3, the complement $S \setminus U$ is a finite union of divisors.

Assume embedded resolution of singularities for reduced subschemes of S of pure codimension 1*.*

Then, upon replacing S by a smooth projective variety with birational morphism to S, we may suppose that the complement of U is a simple normal crossing divisor $D_1 \cup \cdots \cup D_\ell$. Let

$$X := \sqrt[n]{(S, \{D_1, \dots, D_\ell\})},$$

the iterated root stack of S along the divisors D_i. We apply Lemma 2 to the scheme obtained by gluing the local rings at the generic points of the D_i along the generic point of S. Then [3, §VII.5, Thm 5.7] implies that α extends to an open neighborhood of the root stack over this scheme. Hence, by Lemma 3, there is a unique $\beta \in \mathrm{Br}(X)[n]$ that restricts to α.

Now suppose that α is the class of a central simple algebra A of dimension d^2 as a $k(S)$-vector space, with $n \mid d$. Let $\beta_0 \in H^2(X, \mu_n)$ be a lift of $\beta \in \mathrm{Br}(X)[n]$ with corresponding gerbe G_0 banded by μ_n. Now A is the fiber at the generic point of some sheaf of Azumaya algebras $\mathcal{A}$ on an open $W \subset X$ with complement of codimension at least 3 ([10, Thm 2.1]). The Brauer class of the pullback of $\mathcal{A}$ to $W \times_X G_0$ is trivial, hence this pullback is the endomorphism algebra of a locally free coherent sheaf of rank d, which is the restriction of a coherent sheaf $\mathcal{E}_0$ on G_0.

Assume resolution of singularities for reduced Noetherian DM stacks of finite type over k of pure dimension $\dim(S)$*.*

The identity of $\mathcal{E}_0|W \times_X G_0$ induces a morphism to the Grassmannian of rank d quotients of $\mathcal{E}_0$ ([15]). Apply resolution of singularities to the closure of the image to obtain a smooth DM stack Y with a projective morphism to X that restricts to an isomorphism over W, a gerbe $H_0 := Y \times_X G_0$, and a locally free coherent sheaf $\mathcal{F}_0$ on H_0 whose restriction over W is isomorphic to the restriction of $\mathcal{E}_0$. In this situation, let $\gamma_0 := \beta_0|_Y$.

In the proof of Theorem 4 we specialize the above to $n = d = 2$.

Proof. The proof begins with a series of reductions steps, starting with Y as above, equipped with a sheaf of Azumaya algebras, restricting to the quaternion algebra A over $k(S)$, associated with the generic fiber of π.

Step 1. We may suppose that $Y \cong \sqrt{(T, \{E_1, \dots, E_m\})}$ for some smooth projective variety T with birational morphism to S and irreducible divisors E_i such that $E_1 \cup \dots \cup E_m$ is a simple normal crossing divisor and such that at the generic point of each E_i, the projective representation $\mu_2 \to \mathrm{PGL}_2$ given by the sheaf of Azumaya algebras is nontrivial. Indeed, the destackification program ([7, Thm 1.2]) yields a morphism $\widetilde{Y} \to Y$ that is a composition of blow-ups with smooth centers, such that

$$\widetilde{Y} \cong \sqrt{(T, \{E_1, \dots, E_m\})}$$

for a smooth projective variety T and irreducible divisors E_i such that $E_1 \cup \dots \cup E_m$ is a simple normal crossing divisor. We pull back the sheaf of Azumaya algebras to $\widetilde{Y}$. If there is an i such that the projective representation $\mu_2 \to \mathrm{PGL}_2$ over a general point of E_i is trivial, the sheaf of Azumaya algebras descends to

$$\sqrt{(T, \{E_1, \dots, \widehat{E_i}, \dots, E_m\})}.$$

Step 2. We may suppose, additionally, that generically along every component of $E_i \cap E_{i'}$ for $i \neq i'$ the projective representation of $\mu_2 \times \mu_2$ is faithful. Let $F \subset E_i \cap E_{i'}$ be an irreducible component with non-faithful representation. Let $\widetilde{T}$ be the blow-up of T along F. For every $j \in \{1, \dots, m\}$, we denote the proper transform of E_j by $\widetilde{E}_j$, and we denote the exceptional divisor of the blow-up by E'. We let $\widetilde{Y}$ be the normalization of $\widetilde{T} \times_T Y$. Then $\widetilde{Y}$ is isomorphic to the blow-up of Y at the corresponding component of the fiber product of the gerbes of the root stacks, which is itself isomorphic to the root stack

$$\sqrt{(\widetilde{T}, \{\widetilde{E}_1, \dots, \widetilde{E}_m, E'\})}.$$

The projective representation over a general point of E' is trivial, so the sheaf of Azumaya algebras descends to

$$\sqrt{(\widetilde{T}, \{\widetilde{E}_1, \dots, \widetilde{E}_m\})}.$$

Step 3. We may suppose, furthermore, that all triple intersections $E_i \cap E_{i'} \cap E_{i''}$ are empty, where i, i', and i'' are distinct. Since there can never be more than 2 independent commuting subgroups of order 2 in PGL_2 ([6]), the projective representation $(\mu_2)^3 \to PGL_2$ over a general point of $E_i \cap E_{i'} \cap E_{i''}$ has kernel equal to the diagonal μ_2. We blow up T along $E_i \cap E_{i'} \cap E_{i''}$ and proceed as in Step 2.

Step 4. We may suppose, furthermore, that the Brauer class $[A] \in \mathrm{Br}(k(S))$ does not extend across the generic point of E_i for any i. Assume that it does, for some i. Let

$$\widetilde{Y} := \sqrt{(T, \{E_1, \dots, \widehat{E_i}, \dots, E_m\})}.$$

Then by Lemma 3, the Brauer class is restriction of an element $\delta \in \mathrm{Br}(\widetilde{Y})$. Let $\varepsilon \in H^2(\widetilde{Y}, \mu_2)$ denote an arbitrary lift of δ, with corresponding gerbe H_1, such that if we let H_0 denote the base-change

$$Y \times_{\widetilde{Y}} H_1,$$

then on H_0 the sheaf of Azumaya algebras is identified with endomorphism algebra of some locally free coherent sheaf $\mathcal{F}_0$. Notice that H_0 is a root stack over H_1. The relative stabilizer acts with eigenvalues 1 and -1 on fibers of $\mathcal{F}_0$. The (-1)-eigensheaf is a quotient sheaf $\mathcal{L}_{-1}$ supported on the gerbe of the root stack, such that the kernel $\mathcal{F}_1$ in

$$0 \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{L}_{-1} \to 0$$

is again locally free and descends to H_1, yielding a sheaf of Azumaya algebras on $\widetilde{Y}$.

We now have $Y = \sqrt{(T, \{E_1, \dots, E_m\})}$, equipped with a sheaf of Azumaya algebras $\mathcal{A}$, such that the projective representations at a general point of every E_i are nontrivial, the projective representations at a general point of every intersection is faithful, there are no triple intersections, and such that the Brauer class does not extend over any of the generic points of the E_i. Let $P \to Y$ be the smooth $\mathbb{P}^1$-fibration associated with $\mathcal{A}$.

Let T_0 denote the complement of the intersections of pairs of divisors,

$$T_0 := T \setminus \bigcup_{1 \le i < i' \le m} E_i \cap E_{i'}.$$

We apply [12, Proposition 3.1] to $T_0 \times_T P$ to obtain a regular conic bundle

$$\pi_0 \colon V_0 \to T_0.$$

This factors canonically through $\mathbb{P}(\pi_{0*}(\omega^\vee_{V_0/T_0}))$. Let $i : T_0 \to T$ denote the inclusion. We claim that $i_*(\pi_{0*}(\omega^\vee_{V_0/T_0}))$ is a locally free coherent sheaf and, denoting this by $\mathcal{E}$, the closure $V = \overline{V_0}$ of V_0 in $\mathbb{P}(\mathcal{E})$ is a regular conic bundle over T. It suffices to verify these assertions after passing to an algebraic closure of k. Then there is a unique faithful projective representation $\mu_2 \times \mu_2 \to \mathrm{PGL}_2$ (up to conjugacy), cf. [6]. So, by [12, Lemma 2.8] after base change to a suitable affine étale neighborhood $T' = \mathrm{Spec}(B') \to T$ of a given point of an intersection $E_i \cap E_{i'}$, we have

$$Y' \cong \sqrt{(T', \{E_i, E_{i'}\})} = [\mathrm{Spec}(B'[t,t']/(t^2 - x, t'^2 - x'))/\mu_2 \times \mu_2],$$

where x and x' are the respective defining equations for the preimage in T' of E_i and $E_{i'}$, with P' obtained by pulling back $[\mathbb{P}^1/\mu_2 \times \mu_2]$. Here, on $B'[t,t']/(t^2 - x, t'^2 - x')$, the action of the factors of $\mu_2 \times \mu_2$ is by respective scalar multiplication of t and t', while the action on $\mathbb{P}^1$ corresponds to the faithful projective representation $\mu_2 \times \mu_2 \to \mathrm{PGL}_2$. Over T' we compute $\overline{V_0'} \cong \mathrm{Proj}(B'[u,v,w]/(xu^2 + x'v^2 - w^2))$, which is regular.

The fact that the Brauer class does not extend across the generic point of any E_i ensures that the conic bundle $V \to T$ is standard. □

Remark 2. While the general destackification process outlined in [7] requires stacky blow-ups, it is never necessary to take root stacks when all stabilizers are powers of μ_2.

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Paper B: Geometric Brauer Residue via Root Stacks

GEOMETRIC BRAUER RESIDUE VIA ROOT STACKS

JAKOB OESINGHAUS

Abstract. We reinterpret the residue map for the Brauer group of a smooth variety using a root stack construction and Weil restriction for algebraic stacks, and apply the result to find a geometric representative of the residue of the Brauer class associated to a conic bundle.

Contents

1. Introduction

Consider a variety S over a field K, and let n be a positive integer not divisible by the characteristic of K. The *Brauer group* $\mathrm{Br}(S)$ is a classical invariant studied, among other things, in the context of rationality questions, since it is a birational invariant for smooth projective varieties. More precisely, it can be shown that for a *smooth projective* variety X over an algebraically closed field, the n-torsion part of the Brauer group $\mathrm{Br}(X)[n]$ agrees with the group of *unramified* n-torsion elements of the Brauer group of the function field, defined as

$$\bigcap_{R \subset K(X)} \mathrm{Ker}\left(\mathrm{Br}(K(X))[n] \to H^1(\kappa_R, \mathbb{Z}/n\mathbb{Z})\right),$$

where the intersection runs over all (rank one) discrete valuation rings $R \subset K(X)$ such that $K \subset R$, and the map $\mathrm{Br}(K(X))[n] \to H^1(\kappa_R, \mathbb{Z}/n\mathbb{Z})$ is the residue map for this discrete valuation ring.

Let S be a smooth variety over K. Then, for an irreducible divisor S, the residue map ([8, Prop. 2.1])

$$\mathrm{res}\colon \mathrm{Br}(K(S))[n] \to H^1(K(D), \mathbb{Z}/n\mathbb{Z}),$$

measures the ramification of this Brauer group element along the divisor. For example, if $\alpha \in \mathrm{Br}(K(S))[n]$ is the class of a Brauer-Severi scheme of relative dimension $n-1$ over an open subset of S which arises as the restriction of a flat bundle on all of S, then the residue map will yield some information about the degeneration of this bundle along the boundary (cf [3]). In general, the map res is hard to compute explicitly. Root stacks are well-adapted to residue calculations, since in the setting of a discrete valuation ring, every n-torsion Brauer class over the generic point extends uniquely to the n-fold root stack along the closed point.

We use the root stack construction over a discrete valuation ring to reinterpret the residue map in terms of a canonical decomposition of the Brauer group of this root stack (Theorem 1), and use results on Weil restriction for the gerbe of the root stack, which is just $B\mu_n$ over the residue field of the DVR, to give a representative, in terms of a $\mathbb{Z}/n\mathbb{Z}$-torsor, for the residue class in this situation (Proposition 3).

We then apply this result in the geometric situation of a Brauer class associated to a bundle whose generic fiber is a form of projective space. We show that via the operations of Weil restriction and coarse moduli space of an algebraic stack, the residue of the Brauer class along a divisor arises geometrically from the $\mathbb{P}^{n-1}$-bundle associated with the restriction of the Brauer class to the gerbe of the root stack in Proposition 5.

As a special case, we recover a classical result by Artin([3]) using different methods.

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2. The residue map and root stacks

Unless mentioned otherwise, all cohomology groups are étale cohomology groups. We define the (cohomological) *Brauer group* $\mathrm{Br}(X)$ of a Noetherian Deligne-Mumford stack X ([6], [13, IV]) to be the torsion group $H^2(X, \mathbb{G}_m)_{\mathrm{tors}}$.

Let R be a DVR with fraction field K and residue field κ, and let $n > 0$ be an integer not divisible by $\mathrm{char}(\kappa)$. In this situation, by [8, Prop. 2.1], we have the n-torsion residue map

$$\mathrm{res}\colon \mathrm{Br}(K)[n] \to H^1(\mathrm{Spec}(\kappa), \mathbb{Z}/n\mathbb{Z}), \tag{1}$$

which is part of an exact sequence

$$0 \to \mathrm{Br}(R)[n] \to \mathrm{Br}(K)[n] \to H^1(\mathrm{Spec}(\kappa), \mathbb{Z}/n\mathbb{Z}) \to 0.$$

We recall the construction of the map res. Using the Leray spectral sequence for the inclusion $i : \mathrm{Spec}(K) \to \mathrm{Spec}(R)$ and the fact that $R^\ell i_* \mathbb{G}_m[n] = 0$ for $\ell > 0$, one deduces an exact sequence

$$0 \to \mathrm{Br}(R)[n] \to \mathrm{Br}(K)[n] \to H^2(\mathrm{Spec}(R), j_*\mathbb{Z})[n] \to 0,$$

where j is the inclusion of the closed point $\mathrm{Spec}(\kappa)$. Finally, a short exact sequence argument shows that

$$H^2(\mathrm{Spec}(R), j_*\mathbb{Z})[n] = H^2(\mathrm{Spec}(\kappa), \mathbb{Z})[n] = H^1(\mathrm{Spec}(\kappa), \mathbb{Z}/n\mathbb{Z}),$$

which establishes the residue map.

In the same situation as before, we now construct another morphism

$$\mathrm{Br}(K)[n] \to H^1(\mathrm{Spec}(\kappa), \mathbb{Z}/n\mathbb{Z})$$

via root stack methods([1, 5]). Concretely, the n-th root stack of $\mathrm{Spec}(R)$ along $\mathrm{Spec}(\kappa)$ is the Deligne-Mumford stack

$$X = \sqrt[n]{(\mathrm{Spec}(R), \mathrm{Spec}(\kappa))} := [\mathrm{Spec}(R[T]/(T^n - \pi))/\mu_n] \to \mathrm{Spec}(R),$$

which has the property that all n-torsion elements of the Brauer group of K lift to it [11, §3.2]:

$$\mathrm{Br}(K)[n] = \mathrm{Br}(X)[n]. \tag{2}$$

The root stack is an isomorphism over $\mathrm{Spec}(K)$, and the complement of $\mathrm{Spec}(K)$ in the root stack is the *gerbe of the root stack*, a closed substack of X mapping to $\mathrm{Spec}(\kappa)$, which is isomorphic to the classifying stack

$$q : B\mu_{n,\kappa} \to \mathrm{Spec}(\kappa).$$

We can apply the Leray spectral sequence for q and the sheaf $\mathbb{G}_m$, the fact that q has a section, and the fact that the $E_2^{0,2}$-term of this spectral sequence vanishes, to produce a decomposition of $H^2(B\mu_{n,\kappa}, \mathbb{G}_m) = \mathrm{Br}(B\mu_{n,\kappa})$ as

$$\mathrm{Br}(B\mu_{n,\kappa}) \cong \mathrm{Br}(\kappa) \oplus H^1(\mathrm{Spec}(\kappa), \mathbb{Z}/n\mathbb{Z}), \tag{3}$$

where the projection to the first summand arises from the section of q, and the projection to the second summand is the morphism

$$H^2(B\mu_{n,\kappa}, \mathbb{G}_m) \to H^1(\mathrm{Spec}(\kappa), R^1 q_* \mathbb{G}_m) = H^1(\mathrm{Spec}(\kappa), \mathbb{Z}/n\mathbb{Z})$$

arising from the Leray spectral sequence.

3. Main results

Our main theorem shows how the two constructions from the previous section are related.

Theorem 1. *The residue map* (1) *agrees with the composition of the isomorphism* (2)*, restriction to $B\mu_{n,\kappa}$, isomorphism* (3)*, projection to the second factor in the direct sum decomposition, and multiplication by* (-1)*. Concretely, let $\alpha \in \mathrm{Br}(K)[n]$, and extend it to an element $\bar{\alpha} \in \mathrm{Br}(X)[n]$. Then*

$$\mathrm{res}(\alpha) = -\,\mathrm{pr}_2(\bar{\alpha}|_{B\mu_{n,\kappa}}). \tag{4}$$

The proof of Theorem 1 depends on a technical result which forms the basis for our calculation of the right-hand side of (4). We first need to set up some notation. Fix a base field κ. Applying the Leray spectral sequence for the structure morphism of the classifying stacks $B\mu_n$ and $B\mathbb{Z}/n\mathbb{Z}$ over $\mathrm{Spec}(\kappa)$, the cohomology groups $H^1(B\mu_n, \mu_n)$ and $H^1(B\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$

decompose into an arithmetic component (pullback to $\mathrm{Spec}(\kappa)$) and a geometric component (base change to κ^{sep}, where H^1 is identified with group homomorphisms, cyclic of order n, generated by 1_{μ_n}, respectively, $1_{\mathbb{Z}/n\mathbb{Z}}$). We denote by $1_{\mu_n} \boxtimes 1_{\mathbb{Z}/n\mathbb{Z}}$ the cup product

$$pr_1^* 1_{\mu_n} \cup pr_2^* 1_{\mathbb{Z}/n\mathbb{Z}} \in H^2(B(\mu_n \times \mathbb{Z}/n\mathbb{Z}), \mu_n).$$

Lemma 2. *Let κ be a field, and let κ^{sep} be a separable closure of κ. The element*

$$1_{\mu_n} \boxtimes 1_{\mathbb{Z}/n\mathbb{Z}} \in \ker\left(H^2(B(\mu_n \times \mathbb{Z}/n\mathbb{Z}), \mu_n) \to H^2(B\mu_{n,\kappa^{\mathrm{sep}}}, \mu_n)\right) \tag{5}$$

is mapped under the Leray spectral sequence of $p : B(\mu_n \times \mathbb{Z}/n\mathbb{Z}) \to B\mathbb{Z}/n\mathbb{Z}$ to $1_{\mathbb{Z}/n\mathbb{Z}} \in H^1(B\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong H^1(B\mathbb{Z}/n\mathbb{Z}, \mathrm{Hom}(\mu_n, \mu_n))$.

Proof. Since the class (5) vanishes upon pullback to $B\mu_n$, its image in

$$H^1(B\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$$

has vanishing arithmetic component. So we may suppose that κ is separably closed. Then the Leray spectral sequence reduces to the Lyndon-Hochschild-Serre spectral sequence. For the computation we follow the organizational scheme of [14] for the (standard) choices of acyclic resolutions. To obtain $Rp_*\mu_n$ we may exploit the fact that p is obtained from $q : B\mu_n \to \mathrm{Spec}(\kappa)$ by étale base change and push forward the acyclic resolution of μ_n on $B\mu_n$ consisting of μ_n-valued functions on $(\mu_n)^{i+1}$ in degree i for all $i \geq 0$, with homogeneous cochains as μ_n-invariants. By writing these in their inhomogeneous form, $Rq_*\mu_n$, and hence by pullback as well $Rp_*\mu_n$, may be expressed as

$$\mu_n \xrightarrow{0} \bigoplus_{\mu_n} \mu_n \xrightarrow{(c_\beta) \mapsto \left(\frac{c_\beta c_{\beta'}}{c_{\beta\beta'}}\right)} \bigoplus_{\mu_n^2} \mu_n \longrightarrow \dots. \tag{6}$$

For the map to $H^1(B\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ mentioned in the claim, we apply a cutoff functor (cf. [7, 1.4.8]) to obtain the subcomplex

$$\mu_n \xrightarrow{0} \mathrm{Hom}(\mu_n, \mu_n).$$

The inclusion is represented in Figure 1 using an analogous acyclic resolution of $\mathrm{Hom}(\mu_n, \mu_n) = \mathbb{Z}/n\mathbb{Z}$ to that used above, shifted by one (leading to $-d$ in the diagram), and a quasi-isomorphic complex to (6). With this, we may compute the morphism

$$H^2(B\mathbb{Z}/n\mathbb{Z}, [\mu_n \xrightarrow{0} \mathbb{Z}/n\mathbb{Z}]) \to H^2(B\mathbb{Z}/n\mathbb{Z}, Rp_*\mu_n) \cong H^2(B(\mu_n \times \mathbb{Z}/n\mathbb{Z}), \mu_n).$$

By writing down a compatible morphism from the top to the bottom complex in Figure 1 we compute the image of

$$1_{\mathbb{Z}/n\mathbb{Z}} \in H^1(B\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \subset H^2(B\mathbb{Z}/n\mathbb{Z}, [\mu_n \xrightarrow{0} \mathbb{Z}/n\mathbb{Z}]),$$

represented by $(b, b') \mapsto b' - b$ in the group in the top right in Figure 1, to be $1_{\mu_n} \boxtimes 1_{\mathbb{Z}/n\mathbb{Z}}$. □ □

$$\begin{tikzcd}
\mu_n \arrow[r, "0"] \arrow[d, equal] & \mathrm{Map}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \arrow[r, "-d"] \arrow[ddd, dashed, bend left] & \mathrm{Map}((\mathbb{Z}/n\mathbb{Z})^2, \mathbb{Z}/n\mathbb{Z}) \arrow[ddd, dashed, bend left] \\
\mu_n \arrow[r, "0"] \arrow[d, equal] & \mathbb{Z}/n\mathbb{Z} \arrow[r] \arrow[u, "\text{constant maps}"'] \arrow[d, "\mathbb{Z}/n\mathbb{Z} \cong \mathrm{Hom}(\mu_n, \mu_n)"'] & 0 \arrow[u] \arrow[d] \\
\mu_n \arrow[r, "0"] \arrow[d, "\text{constant}"'] & \bigoplus_{\mu_n} \mu_n \arrow[r] \arrow[d, "(c_\beta) \mapsto (\text{constant } c_\beta)_{(\beta, b)}"'] & \bigoplus_{(\mu_n)^2} \mu_n \arrow[d, "(c_{\beta,\beta'}) \mapsto (\text{constant } c_{\beta,\beta'})_{((\beta,b),(\beta',b'))}"'] \\
\mathrm{Map}(\mathbb{Z}/n\mathbb{Z}, \mu_n) \arrow[r] & \bigoplus_{\mu_n \times \mathbb{Z}/n\mathbb{Z}} \mathrm{Map}(\mathbb{Z}/n\mathbb{Z}, \mu_n) \arrow[r] & \bigoplus_{(\mu_n \times \mathbb{Z}/n\mathbb{Z})^2} \mathrm{Map}(\mathbb{Z}/n\mathbb{Z}, \mu_n)
\end{tikzcd}$$

FIGURE 1. The morphisms of complexes giving rise to the (dotted) morphism in the derived category.

Theorem 1. We can assume without loss of generality that R is Henselian. Indeed, if $R \to R^h$ is the Henselization of R, the residue fields of R and R^h are equal, and since the Leray spectral sequence is functorial, the natural diagrams for the residue maps commute. Similarly, due to the nature of the root stack construction, the map on the right-hand side of (4) is functorial for the Henselization.

If R is Henselian, by [13, Rem. III.3.11], we have

$$H^1(\mathrm{Spec}(R), \mathbb{Z}/n\mathbb{Z}) = H^1(\mathrm{Spec}(\kappa), \mathbb{Z}/n\mathbb{Z}) \qquad \text{and} \qquad \mathrm{Br}(R) = \mathrm{Br}(\kappa).$$

We will keep using these isomorphisms implicitly. For elements of $\mathrm{Br}(R)[n] \subset \mathrm{Br}(K)[n]$, both sides of (4) are zero: $\mathrm{Br}(R)[n]$ is the kernel of the residue map, and elements of $\mathrm{Br}(R)[n] = \mathrm{Br}(\kappa)[n]$ end up in the first summand of the decomposition (2). Hence, it suffices to verify the equality (4) for a subset of elements of $\mathrm{Br}(K)[n]$ whose residues attain all elements of $H^1(\mathrm{Spec}(\kappa), \mathbb{Z}/n\mathbb{Z})$.

Pick a uniformizer $\pi \in R$. The morphism $\mu_n \otimes \mathbb{Z}/n\mathbb{Z} \to \mu_n$ induces a cup product pairing

$$\cup \colon H^1(\mathrm{Spec}(K), \mu_n) \otimes H^1(\mathrm{Spec}(K), \mathbb{Z}/n\mathbb{Z}) \to H^2(\mathrm{Spec}(K), \mu_n) \to \mathrm{Br}(K)[n].$$

As our set of elements on which we will verify (4), we choose those of the form $\theta \cup \gamma$, where θ is the class of $K(\pi^{1/n})/K$ and γ is a class in $H^1(\mathrm{Spec}(R), \mathbb{Z}/n\mathbb{Z})$, with the same symbol used to denote its restriction to K.

Recall that root stack X is the quotient

$$X = \sqrt[n]{(\mathrm{Spec}(R), \mathrm{Spec}(\kappa))} = [\mathrm{Spec}(R[T]/(T^n - \pi))/\mu_n],$$

where μ_n acts by scalar multiplication on T.

Fix $\gamma \in H^1(\mathrm{Spec}(R), \mathbb{Z}/n\mathbb{Z})$, which corresponds to a cyclic degree n étale R-algebra S. Then

$$\mathrm{Spec}(S[T]/(T^n - \pi)) \to \mathrm{Spec}(R[T]/(T^n - \pi)) \to X$$

is a $(\mu_n \times \mathbb{Z}/n\mathbb{Z})$-torsor over X. For this torsor, the extension $\bar{\alpha} \in \mathrm{Br}(X)[n]$ is the image under $H^2(X, \mu_n) \to \mathrm{Br}(X)[n]$ of the cup product class $\theta \cup \gamma$ to X.

The extensions of θ, γ, and $\theta \cup \gamma$ to the root stack are pulled back from the classifying space $B(\mu_n \times \mathbb{Z}/n\mathbb{Z})$ via the morphism

$$X \to B(\mu_n \times \mathbb{Z}/n\mathbb{Z}). \tag{7}$$

corresponding to this $(\mu_n \times \mathbb{Z}/n\mathbb{Z})$-torsor.

In the following, we use notation taken from [13, Exa. III.2.6] to write down a representation of $\bar{\alpha} \in \mathrm{Br}(X)[n]$ as a Čech 2-cocycle, by transforming the simplicial Čech associated to (3). We make the identification of $(\mu_n \times \mathbb{Z}/n\mathbb{Z})$-torsors

$$\mathrm{Spec}(S[T]/(T^n - \pi)) \times_X \mathrm{Spec}(S[T]/(T^n - \pi)) = \mathrm{Spec}(S[T]/(T^n - \pi)) \times (\mu_n \times \mathbb{Z}/n\mathbb{Z}),$$

denoting an element of the μ_n-factor by β and an element of the $\mathbb{Z}/n\mathbb{Z}$-factor by b; analogously, we use pairs of such elements for the triple fiber product over X.

A representative for $\bar{\alpha}$ in this notation is

$$(\beta^{b'})_{((\beta,b),(\beta',b'))}.$$

We define a $\mathbb{G}_m$-valued Čech 2-cocycle $\varepsilon_{b,b'}$ depending (only) on $b, b' \in 0, \ldots, n-1$:

$$\varepsilon_{b,b'} := \begin{cases} 1 & \text{if } b + b' < n, \\ \pi^{-1} & \text{if } b + b' \geq n. \end{cases}$$

The difference between these cocyles is a coboundary; indeed, the coboundary of the 1-cochain $(\pi^{b/n})_{(\beta,b)}$ is $(\varepsilon_{b,b'}^{-1}\beta^{b'})_{((\beta,b),(\beta',b'))}$. Since the representative ε is independent of β, it is pulled back from the étale cover $\mathrm{Spec}(K \otimes_R S) \to \mathrm{Spec}(K)$. We can explicitly compute the residue of the class represented by the Čech 2-cocyle which ε is pulled back from, and we find it to be $-\gamma|_{\mathrm{Spec}(\kappa)}$.

It remains to be shown that the right-hand side of (4) yields the same element. To show this, we compute on $B(\mu_n \times \mathbb{Z}/n\mathbb{Z})$ using Lemma 2 and pull back the result via the map (7). □ □

For the next result, we will need the notion of restriction of scalars f_* for a proper flat finitely presented morphism f of algebraic stacks with finite diagonal, as described in [9].

Proposition 3. *Let $\mathcal{G} \to B\mu_{n,\kappa}$ be the gerbe banded by μ_n whose class $\tilde{\alpha} \in H^2(B\mu_{n,\kappa}, \mu_n)$ is the unique lift of a Brauer class $\alpha \in \mathrm{Br}(B\mu_{n,\kappa})[n]$ such that the pullback of $\tilde{\alpha}$ to $B\mu_{n,\bar{\kappa}}$ vanishes. With notation $q\colon B\mu_{n,\kappa} \to \mathrm{Spec}(\kappa)$*

for the structure morphism and [ ] *for coarse moduli space,* $[q_*\mathcal{G}]$ *is a* $\mathbb{Z}/n\mathbb{Z}$*-torsor whose class is* $-\mathrm{pr}_2(\alpha)$*.*

Proof. Adding an element of $\mathrm{Br}(\kappa)[n]$ to α does not change $[q_*\mathcal{G}]$. So it suffices to treat the case that α restricts to 0 in $\mathrm{Br}(\kappa)$. Then we reduce as before to the computation in the universal case, that is, over $B(\mu_n \times \mathbb{Z}/n\mathbb{Z})$, and again reduce to carrying out the computation when κ is separably closed. Let Γ be the subgroup of $GL_n(\kappa)$ generated by the scalar nth roots of unity, the permutation matrix for the n-cycle $(1, 2, \dots, n)$ and a diagonal matrix whose entries are successive powers of a primitive nth root of unity. Then Γ is a central μ_n-extension of $\mu_n \times \mathbb{Z}/n\mathbb{Z}$, where if we take $\Gamma \to \mu_n$ defined for $A \in \Gamma$ by the ratio of successive nonzero entries of A, and $\Gamma \to \mathbb{Z}/n\mathbb{Z}$, by the position of the nonzero entry in the first row of A, the class in $H^2(\mu_n \times \mathbb{Z}/n\mathbb{Z}, \mu_n)$ of the group extension

$$1 \to \mu_n \to \Gamma \to \mu_n \times \mathbb{Z}/n\mathbb{Z} \to 1$$

is that of the 2-cocycle

$$(\beta'^b)_{((\beta,b),(\beta',b'))},$$

i.e., is $-1_{\mu_n} \boxtimes 1_{\mathbb{Z}/n\mathbb{Z}} \in H^2(\mu_n \times \mathbb{Z}/n\mathbb{Z}, \mu_n)$.

With $p\colon B(\mu_n \times \mathbb{Z}/n\mathbb{Z}) \to B\mathbb{Z}/n\mathbb{Z}$ as above, the relative moduli space [2, §3] of $p_*B\Gamma$ is the universal $\mathbb{Z}/n\mathbb{Z}$-torsor $\mathrm{Spec}(\kappa) \to \mathbb{Z}/n\mathbb{Z}$. Comparing with the computation above, we obtain the result. □ □

We make a definition analogous to Brauer-Severi schemes for the case that the base is an algebraic stack.

Definition 1. A *Brauer-Severi stack of relative dimension* $n-1$ over an algebraic stack S is a smooth, proper, representable morphism $p : P \to S$ such that all geometric fibers of p are projective spaces $\mathbb{P}^{n-1}$.

A Brauer-Severi stack over a scheme S is clearly just a Brauer-Severi scheme over S. The following is an adaptation of a definition that was used in [10] in the case $n = 3$.

Definition 2. Let κ be a field, and let n be a positive integer not divisible by $\mathrm{char}(\kappa)$. Let P be a Brauer-Severi variety of dimension $n-1$ over κ. We say that an action of μ_n on P is *balanced* if, after passing to $\bar\kappa$ and identifying $P_{\bar\kappa} \simeq \mathbb{P}^{n-1}_{\bar\kappa}$, the action of $\beta \in \mu_n$ is given by

$$(x_1 : x_2 : \cdots : x_n) \mapsto (\beta x_1 : \beta^2 x_2 : \cdots : x_n).$$

We make an analogous definition for a Brauer-Severi stack of relative dimension $n-1$ over $B\mu_{n,\kappa}$.

Recall that we denote the structure morphism $B\mu_{n,\kappa} \to \mathrm{Spec}(\kappa)$ by q.

Proposition 4. *Suppose that α is the Brauer class of a PGL_n-torsor over $B\mu_{n,\kappa}$ whose associated Brauer-Severi stack $P \to B\mu_{n,\kappa}$ is balanced. We endow the coarse moduli space of the Weil restriction $[q_*P]$ the structure of a $\mathbb{Z}/n\mathbb{Z}$-torsor, given by translation on characters of eigenspaces for local* *choices of rank n vector bundles E with $P \cong \mathbb{P}(E)$. Then the associated class in $H^1(\mathrm{Spec}(\kappa), \mathbb{Z}/n\mathbb{Z})$ is inverse to that of $[q_*\mathcal{G}]$, where $\mathcal{G}$ is as in Proposition 3.*

Proof. To compute this class, we make a base change to $\mathcal{G}$. On $\mathcal{G}$, there is a global choice of rank n vector bundle E with $P \cong \mathbb{P}(E)$. Any character of μ_n induces a section of $[q_*\mathbb{P}(E)] \to [q_*\mathcal{G}]$, given by eigenspaces of this character. If we endow $[q_*P]$ with the inverse of the structure of a $\mathbb{Z}/n\mathbb{Z}$-torsor given by translation on characters of eigenspaces, the composite

$$[q_*\mathcal{G}] \to [q_*P]$$

is an equivariant isomorphism. □ □

4. Application to Brauer-Severi schemes

We can now apply the results of the previous section to the geometric situation. Given a Brauer class as in Propositon 4, we can combine Propositions 3 and 4 to allow us to identify the class of the $\mathbb{Z}/n\mathbb{Z}$-torsor $[q_*P]$ with $\mathrm{pr}_2(\alpha)$. This leads to the following result.

Proposition 5. *Let R be a DVR with fraction field K and residue field κ, let $n > 0$ be an integer not divisible by $\mathrm{char}(\kappa)$, and let $\alpha \in \mathrm{Br}(K)[n]$, with extension to $\bar{\alpha} \in \mathrm{Br}(X)[n]$, where $X = \sqrt[n]{(\mathrm{Spec}(R), \mathrm{Spec}(\kappa))} \to \mathrm{Spec}(R)$. If the restriction of $\bar{\alpha}$ to the gerbe of the root stack $B\mu_{n,\kappa}$ is the Brauer class of a PGL_n-torsor whose associated Brauer-Severi stack $P \to B\mu_{n,\kappa}$ is balanced, then the $\mathbb{Z}/n\mathbb{Z}$-torsor $[q_*P]$, where q denotes the structure morphism $B\mu_{n,\kappa} \to \mathrm{Spec}(\kappa)$, has the class $\mathrm{pr}_2(\alpha) \in H^1(\mathrm{Spec}(\kappa), \mathbb{Z}/n\mathbb{Z})$, in the notation of Theorem 1.*

As a special case of our results, we recover some classical observations due to Artin ([3, Thm 1.4], cf. the more recent [4]). As an example, consider the residue map for a standard conic bundle $Q \to S$ (cf. [15] for a definition). Let $\alpha \in \mathrm{Br}(k(S))[2]$ be the Brauer class of the bundle. Let D be an irreducible divisor of S such that α ramifies along D. Since the conic bundle is standard, we can apply Proposition 5 to the local ring R at the generic point of D. Using the notation of the proposition, we have that $\bar{\alpha}$ is the class of the smooth conic bundle P over all of the root stack, which extends the given smooth bundle over the generic point of $\mathrm{Spec}(R)$. Hence, by comparing it to the singular fiber of the bundle over the generic point of D, we conclude that the residue of α along D can be seen geometrically as the class of the space of components over its generic point.

An analogous observation is true for standard Brauer-Severi surface bundles ([12]).

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Paper C: Quasi-symmetric Functions and the Chow Ring of the Stack of Expanded Pairs

QUASI-SYMMETRIC FUNCTIONS AND THE CHOW RING OF THE STACK OF EXPANDED PAIRS

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Abstract. We show that the Hopf algebra of quasi-symmetric functions arises naturally as the integral Chow ring of the algebraic stack of expanded pairs originally described by J. Li, using a more combinatorial description in terms of configurations of line bundles. In particular, we exhibit a gluing map which gives rise to the comultiplication. We then apply the result to calculate the Chow rings of certain stacks of semistable curves.

Contents

1. Introduction

In order to study relative Gromov-Witten invariants of a pair (X, D) of a scheme X over $\mathbb{C}$ and a Cartier divisor $D \subset \mathbb{C}$, J. Li introduced the notion of a family of *expansions* of the pair (X, D) (cf. [Li01, Li02]). An expansion of the pair (X, D) of length ℓ is constructed by gluing ℓ copies of the projectivized normal bundle of D to X:

$$X(\ell) := X \sqcup_D \underbrace{P \sqcup_D \cdots \sqcup_D P}_{\ell \text{ times}},$$

where $P = \mathbb{P}(\mathcal{O}_X(D)|_D \oplus \mathcal{O}_D)$. Since P carries a $\mathbb{G}_m$-action by scaling, an expansion of length ℓ comes with an action of $\mathbb{G}_m^\ell$.

Let $\mathcal{A} := [\mathbb{A}/\mathbb{G}_m]$ be the stack quotient of the affine line by the standard action of the multiplicative group; it is the moduli space of pairs $(\mathcal{L}, s)$ of a line bundle $\mathcal{L}$ and a section s of $\mathcal{L}$. Let $\mathcal{D} := [0/\mathbb{G}_m] \cong \mathbb{G}_m$ be the vanishing locus of the universal section. The pair $(\mathcal{A}, \mathcal{D})$ forms the universal pair of an algebraic stack and a Cartier divisor on it. In [ACFW13], the authors used this fact to define a stack $\mathcal{T}$ of expansions of any expansions of the universal pair $(\mathcal{A}, \mathcal{D})$, proved that it is an algebraic stack locally of finite type. The same object has also been studied in [GV05].

The stack $\mathcal{A}$ and its powers $\mathcal{A}^n \cong [\mathbb{A}^n/\mathbb{G}_m^n]$ are connected to logarithmic geometry; in fact, they form an open substack of the stack of logarithmic structures ([Ols03]). It is possible to adopt the logarithmic point of view to identify $\mathfrak{T}$ as the stack of aligned log structures (cf [ACFW13, 8] and [BV12]). This allows a more combinatorial description of $\mathfrak{T}$ as a colimit of the $\mathcal{A}^n$ by étale morphisms.

The Hopf algebra QSym of quasi-symmetric functions is well-studied object that arises as a generalization of symmetric functions. As an algebra, it is commutative and graded, and it is free[1] over $\mathbb{Z}$ with finitely many generators in each degree, though writing down explicit integral generators is not straightforward (cf [Haz10]). The coalgebra structure is not cocommutative. It is straightforward to see that QSym arises as a certain projective limit of polynomial rings in the category of graded algebras.

We prove that QSym arises as the Chow ring of $\mathfrak{T}$. To be more precise, we calculate the Chow ring of $\mathcal{X} \times \mathfrak{T}$ for a smooth algebraic stack $\mathcal{X}$ of finite type over the base field, and show that the colimit construction of $\mathfrak{T}$ gives rise to an isomorphism

$$\mathrm{CH}^\bullet(\mathcal{X} \times \mathfrak{T}) \cong \mathrm{CH}^\bullet(\mathcal{X}) \otimes \mathrm{QSym}$$

in Theorem 3. We then show that there exists an étale, but non-separated morphism $\mathfrak{T} \times \mathfrak{T} \to \mathfrak{T}$ which exhibits $\mathfrak{T}$ as a monoid object and induces the comultiplication of QSym on the level of Chow cohomology.

In section 5, we use the fact that $\mathfrak{T}$ has an interpretation as an open substack of the moduli stack of 3-pointed semistable curves $\mathfrak{M}_{0,3}^{ss}$ to calculate the intersection rings of $\mathfrak{M}_{0,3}^{ss}$. Moreover, the stack $\mathfrak{M}_{0,2}^{ss}$ can be seen as a non-rigid variant of $\mathfrak{T}$, which allows us to compute its Chow ring, and the action of the natural involution on this ring, as a corollary of our computations for $\mathfrak{T}$. Our study was partially motivated by Chow group calculations for an open substack of the stack of unpointed curves $\mathfrak{M}_0$ due to Fulghesu ([Ful10]).

Notation. Throughout this note, we let k be any separably closed base field. Let $\mathfrak{C}$ be the category of finite ordered sets and order-preserving injections, and we let $\mathfrak{C}_{\leq m}$ be the full subcategory consisting of sets with m or less elements.

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[1] I.e., isomorphic to a polynomial algebra.

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2. Quasisymmetric functions

Quasisymmetric functions are a generalization of the well-known Hopf algebra of symmetric functions due to Gessel ([Ges84]). Since we will identify the intersection ring of the stack of expanded pairs as the ring of quasisymmetric functions, we first take some time to describe this ring in more detail. All of the material in this section is classical, except for possibly Proposition 5, which we did not find in the literature. A very in-depth treatment of quasisymmetric functions can be found in [LMvW13, GR14].

2.1. Definitions.

Given any totally ordered set $\mathfrak{I}$, we can consider the commutative graded ring

$$\mathbb{Z}[\![\alpha_i]\!]_{i\in\mathfrak{I}} \tag{1}$$

of formal power series in the variables $\{\alpha_i\}_{i\in\mathfrak{I}}$ and with $\mathbb{Z}$-coefficients.

Definition 1. The algebra of *quasisymmetric functions on the index set* $\mathfrak{I}$ is the subring $\mathrm{QSym}_{\mathfrak{I}} \subset \mathbb{Z}[\![\alpha_i]\!]_{i\in\mathfrak{I}}$ consisting of power series f of bounded degree satisfying the following condition:

For every two increasing sequences $i_1 < \cdots < i_\ell$ and $j_1 < \cdots < j_\ell$ of elements of $\mathfrak{I}$ of length ℓ, and for every $(I_1, \ldots, I_\ell) \in \mathbb{Z}^\ell_{>0}$, the coefficients of f for the monomials $\alpha_{i_1}^{I_1} \cdots \alpha_{i_\ell}^{I_\ell}$ and $\alpha_{j_1}^{I_1} \cdots \alpha_{j_\ell}^{I_\ell}$ are equal.

It is also possible to define quasisymmetric functions with R-coefficients for any commutative ring R as $\mathrm{QSym}_{\mathfrak{I}} \otimes R$.

Notation. We will usually consider the index set $\mathfrak{I} = \mathbb{Z}_{>0}$, and write $\mathrm{QSym} := \mathrm{QSym}_{\mathbb{Z}_{>0}}$. We also define $\mathrm{QSym}_n := \mathrm{QSym}_{[n]}$, where $[n] := \{1 < \cdots < n\}$.

Definition 2. Let $n \in \mathbb{N}$. A *composition* I of n (of length $\ell(I) := \ell$) is an ordered ℓ-tuple $(I_1, \ldots, I_\ell) \in \mathbb{Z}_{>0}$ of positive integers such that $I_1 + \cdots + I_\ell = n$. Let Comp be the set of all compositions, and let Comp_n be the set of compositions of n. Given two compositions $I = (I_1, \ldots, I_k)$ and $J = (J_1, \ldots, J_\ell)$, we write

$$I \cdot J := (I_1, \ldots, I_k, J_1, \ldots, J_\ell).$$

There is a natural basis of $\mathrm{QSym}_{\mathfrak{I}}$ called the *monomial basis*, indexed by compositions. For a composition I, let M_I be the monomial

$$M_I := \sum_{i_1<\cdots<i_\ell} \alpha_{i_1}^{I_1} \cdots \alpha_{i_\ell}^{I_\ell}. \tag{2}$$

Then the monomials M_I form a basis of $\mathrm{QSym}_{\mathfrak{I}}$ if $\mathfrak{I}$ is infinite, and the monomials indexed by compositions I with $\ell(I) \leq |\mathfrak{I}|$ form a basis of $\mathrm{QSym}_{\mathfrak{I}}$ if $\mathfrak{I}$ is finite[2]. In particular, this implies that $\mathrm{QSym}_{\mathfrak{I}}$ does not depend on the index set as long as $\mathfrak{I}$ is infinite. Nonetheless, keeping track of indices can often be useful.

It is clear that $\deg(M_I) = n$ if I is a composition of n, hence compositions of n give rise to a basis for the degree n part of QSym. The assignment $\mathfrak{I} \mapsto \mathrm{QSym}_{\mathfrak{I}}$

[2] All M_I for $\ell(I) > |\mathfrak{I}|$ are identically 0.

defines a contravariant functor $\mathfrak{QSym}$ from the category of totally ordered sets and order-preserving injections to the category of commutative graded rings; a morphism $g : \mathfrak{I} \hookrightarrow \mathfrak{J}$ maps to the "evaluation" homomorphism setting all α_j for $j \notin g(\mathfrak{I})$ to 0. This functor behaves nicely in the monomial basis:

Proposition 1. *For any order-preserving injection $g : \mathfrak{I} \hookrightarrow \mathfrak{J}$ and any composition I, we have*

$$\mathfrak{QSym}(g)(M_I) = M_I.$$

In particular, if $\mathfrak{I}$ is finite, then the kernel of $\mathfrak{QSym}(g)$ is generated by monomials M_I for compositions I satisfying $\ell(I) > |\mathfrak{I}|$, and the restriction of $\mathfrak{QSym}(g)$ to the subalgebra generated by monomials M_I for $\ell(I) \leq |\mathfrak{I}|$ is an isomorphism. □

Note that there also exists a covariant functor from the category of totally ordered sets and order-preserving injections to the category of graded $\mathbb{Z}$-modules, sending $M_I \to M_I$; however, these inclusions of $\mathbb{Z}$-modules are not algebra homomorphisms. For example, for the inclusion $\{1\} \hookrightarrow \{1 < 2\}$, the product $M_{(1)} \cdot M_{(1)}$ is equal to $M_{(2)}$ in the source, but it is $M_{(2)} + 2M_{(1,1)}$ in the target.

Example. Let us list all monomials of degree up to 3, with index set $\mathbb{Z}_{>0}$.

- $M_\emptyset = 1$.
- $M_{(1)} = \sum_i x_i = x_1 + x_2 + \dots$.
- $M_{(2)} = \sum_i x_i^2 = x_1^2 + x_2^2 + \dots$.
- $M_{(1,1)} = \sum_{i<j} x_i x_j = x_1x_2 + x_1x_3 + x_2x_3 + \dots$.
- $M_{(3)} = \sum_i x_i^3 = x_1^3 + x_2^3 + \dots$.
- $M_{(2,1)} = \sum_{i<j} x_i^2 x_j = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + \dots$.
- $M_{(1,2)} = \sum_{i<j} x_i x_j^2 = x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + \dots$.
- $M_{(1,1,1)} = \sum_{i<j<k} x_ix_jx_k = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + \dots$.

2.2. **Free generators for the multiplication.** There is a description of the multiplication of monomials in purely combinatorial terms.

Proposition 2 ([GR14, Prop 5.3])**.** *Fix an infinite index set $\mathfrak{I}$, and $\ell, m \in \mathbb{Z}_{>0}$. Fix two pairwise disjoint chain posets $\{i_1 < \dots < i_\ell\}$ and $\{j_1 < \dots < j_m\}$.*

For compositions $I = (I_1, \dots, I_\ell)$ and $J = (J_1, \dots, J_m)$, we have

$$M_I \cdot M_J = \sum_f M_{\mathrm{wt}\, f},$$

where the sum runs over all surjective, strictly order-preserving maps

$$\{i_1, \dots, i_\ell\} \sqcup \{j_1, \dots, j_m\} \to \{1, \dots, n\}$$

for some $n \in \mathbb{N}$, and the composition $\mathrm{wt}\, f$ is defined by

$$(\mathrm{wt}\, f)_x := \sum_{i_u \in f^{-1}(x)} I_u + \sum_{j_u \in f^{-1}(x)} J_u.$$

As a consequence of Proposition 1 and Proposition 2, we can perform additive computations involving only monomials of length $\leq \ell$ using any index set with ℓ elements, and we can compute the product of two monomials of degree ℓ and m using any index set with $\ell + m$ elements.

Example. We can compute the product $M_{(1,2)} \cdot M_{(1,1)}$ as follows, using Proposition 2.

$$\begin{aligned} M_{(1,2)} \cdot M_{(1,1)} =& M_{(1,2,1,1)} + 2M_{(1,1,2,1)} + 3M_{(1,1,1,2)} \\ &+ M_{(2,2,1)} + M_{(1,3,1)} + M_{(2,1,2)} + 2M_{(1,1,3)} + M_{(1,2,2)} \\ &+ M_{(2,3)}. \end{aligned}$$

It can be shown that quasisymmetric functions form a polynomial algebra, i.e., there is a subset generating QSym as an algebra, without any relations. To write down a more precise statement, we need to introduce some notation.

Definition 3. The *lexicographic order* is the total order on Comp defined by $I \leq J$ if

- either there exists an $i \leq \min\{\ell(I), \ell(J)\}$ such that $I_i < J_i$ and for every $j < i$, we have $I_j = J_j$,
- or I is a prefix of J, i.e. we can write $J = I \cdot K$ for another composition K.

Definition 4. We say that a nonempty composition I is a *Lyndon composition* if every nonempty proper suffix of I is greater than I. Concretely, this means that whenever we can write $I = J \cdot K$ for nonempty compositions J and K, we have $K > I$. Let $\mathfrak{L}$ be the set of Lyndon compositions.

Example. Every composition of the form (a) is Lyndon. A composition of the form (a, b) is Lyndon if and only if $b > a$. A composition of the form (a, b, c) is Lyndon if and only if $c > a$ and $b \geq a$.

Let μ be the number-theoretic Möbius function, i.e. $\mu(d)$ is the sum of the d-th primitive roots of unity.

Proposition 3 ([Wit37])**.** *The number b_n of Lyndon compositions of length n equals*

$$b_n := \frac{1}{n} \sum_{d|n} \mu(d) \left(2^{n/d} - 1\right).$$

Hence, the numbers b_n satisfy $\sum_{d|n} d b_d = 2^n - 1$.

Remark. The sequence b_n starts as follows (sequence A059966 in OEIS):

$$(b_1, b_2, \dots) = (1, 1, 2, 3, 6, 9, 18, \dots).$$

Theorem 1 ([Haz01, Haz10])**.** QSym *is isomorphic to a graded polynomial ring with b_n generators in degree n.*

We can write down a set of free rational generators quite explicitly. This is also possible over the integers, but requires a larger notational effort[3]. In fact, the morphism

$$\begin{aligned} \mathbb{Q}[x_I]_{I \in \mathfrak{L}} \to& \mathrm{QSym} \otimes \mathbb{Q} \\ x_I \mapsto& M_I \end{aligned}$$

is an isomorphism of graded algebras, for $\deg x_I = |I|$.

[3]To see how an integral basis can be constructed with the same index set, see [Haz10], or [GR14, 6.5] for a more detailed explanation.

Example. The generators corresponding to Lyndon compositions up to degree 4 are as follows:

$$M_{(1)}, M_{(2)}, M_{(3)}, M_{(1,2)}, M_{(4)}, M_{(1,3)}, M_{(1,1,2)}.$$

2.3. **Hopf algebra structure.** The comultiplication for quasisymmetric functions extend the one for symmetric functions. In the monomial basis, it can be described as follows. Let

$$\begin{aligned} \Delta\colon \mathrm{QSym} &\to \mathrm{QSym} \otimes \mathrm{QSym} \\ M_I &\mapsto \sum_{I = J \cdot K} M_J \otimes M_K \end{aligned} \qquad \begin{aligned} \varepsilon\colon \mathrm{QSym} &\longrightarrow \mathbb{Z} \\ n + \sum_{|I| \geq 1} a_I M_I &\mapsto n. \end{aligned}$$

and

$$\begin{aligned} S\colon \mathrm{QSym} &\to \mathrm{QSym} \\ M_I &\mapsto (-1)^{\ell(I)} \sum_{\substack{J \text{ coarser than} \\ \mathrm{rev}(I)}} M_J, \end{aligned}$$

where the reverse $\mathrm{rev}(I)$ of a composition $I = (I_1, \ldots, I_\ell)$ is $\mathrm{rev}(I) = (I_\ell, \ldots, I_1)$ and J is said to be *coarser* than K if J can be obtained by successively summing some of the adjacent entries of K.

Proposition 4. *The triple* $(\mathrm{QSym}, \Delta, \varepsilon)$ *defines a coassociative coalgebra structure on* QSym *which is compatible with the algebra structure. Moreover, the bialgebra* QSym *is a graded Hopf algebra with antipode* S. ☐

Example. Consider the monomial $M := M_{(3,1,4)}$. Then

$$\Delta(M) = M_{(3,1,4)} \otimes 1 + M_{(3,1)} \otimes M_{(4)} + M_{(3)} \otimes M_{(1,4)} + 1 \otimes M_{(3,1,4)}$$

and

$$S(M) = -\left(M_{(4,1,3)} + M_{(5,3)} + M_{(4,4)} + M_{(8)}\right).$$

2.4. **Quasisymmetric functions as a limit.** There is a construction of quasisymmetric functions as a categorical limit, which is useful for our purposes. Recall that $\mathfrak{C}$ denotes the category of finite ordered sets and order-preserving injections. The assignment

$$S \mapsto \mathbb{Z}[\alpha_s]_{s \in S} \qquad (\phi : S \to T) \mapsto g_\phi,$$

where

$$g_\phi(x_t) = \begin{cases} \alpha_s, & \text{if there exist some } s \in S \text{ such that } \phi(s) = t, \\ 0, & \text{otherwise.} \end{cases}$$

defines a contravariant functor from $\mathfrak{C}$ to the category of graded rings.

Proposition 5. *For every finite ordered set* S*, consider the restriction morphism* $\mathrm{QSym} \to \mathrm{QSym}_S$ *from Proposition 1 for any order-preserving injection* $S \hookrightarrow \mathbb{Z}$*. Then* QSym*, together with this family of restriction morphisms, satisfies the universal property of the limit*

$$\varprojlim_{\mathfrak{C}} \mathbb{Z}[\alpha_s]_{s \in S}.$$

Proof. For simplicity, we replace $\mathfrak{C}$ by the equivalent full subcategory consisting only of the objects $[n] := \{1 < \cdots < n\}$ for $n \in \mathbb{N}$. Note that the morphisms are generated by the family of morphisms $j_i^n : [n] \to [n+1]$ for $i \in \{1, \ldots, n+1\}$, where j_i^n is the unique map whose image does not contain i. Denote by

$$\pi_n : \mathrm{QSym} \to \mathrm{QSym}_{[n]} \hookrightarrow \mathbb{Z}[\alpha_i]_{i=1,\ldots,n}$$

the projections, and let $g_i^n := g_{j_i^n}$. It is clear from the definition that the projections π_n satisfy the necessary compatibility conditions, since they preserve monomials.

To prove that QSym satisfies the universal property, let now $R^\bullet$ be any graded ring, and assume we are given morphisms of graded rings

$$f_n : R^\bullet \to \mathbb{Z}[\alpha_1, \ldots, \alpha_n]$$

such that $g_i^n \circ f_{n+1} = f_n$, then for any $r \in R^d$ and every $1 \leq i \leq n+1$, we have

$$f_{n+1}(r)(\alpha_1, \ldots, \alpha_{i-1}, 0, \alpha_i, \ldots, \alpha_n) = f_n(r)(\alpha_1, \ldots, \alpha_n). \tag{3}$$

This implies that for indices $i_1 < \cdots < i_s$ and $j_1 < \cdots < j_s$ and exponents $k_1 + \cdots + k_s = d$, the coefficient of $\alpha_{i_1}^{k_1} \cdots \alpha_{i_s}^{k_s}$ and the coefficient of $\alpha_{j_1}^{k_1} \cdots \alpha_{j_s}^{k_s}$ in the homogeneous degree d polynomial $f_{n+1}(r)$ agree. This is proved by iterating the above equalities (3), inserting zero everywhere except for indices in $\{i_1, \ldots, i_s\}$, respectively $\{j_1, \ldots, j_s\}$. We conclude that $f_n(r)$ is a quasisymmetric function of degree d in n variables, for every n. Moreover, the value $f_e(r)$ for $e \geq 0$ is uniquely determined by $f_d(r)$, because a quasisymmetric function of degree d in any number of variables is determined by its coefficients for monomials in the first d variables, and because $f_0(r), \ldots, f_{d-1}(r)$ are determined by (3). Hence, we can define $\psi(r) \in A_d$ by extending the quasisymmetric function $f_d(r)$ to a countable number of variables. By construction, we have $\pi_n \circ \psi = f_n$. The lifting ψ is unique, because the map $\pi_d|_{A_d}$ is injective. □

Remark. With the same proof, we can also conclude that

$$\varprojlim_{\mathfrak{C}_{\leq m}} \mathbb{Z}[\alpha_s]_{s \in S} \cong \mathrm{QSym}_m.$$

3. Definitions and setup

We denote the Chow groups of an algebraic stack $\mathcal{X}$ by $\mathrm{CH}_\bullet(\mathcal{X})$, and the Chow ring of a smooth, equidimensional algebraic stack by

$$\mathrm{CH}^\bullet(\mathcal{X}) = \mathrm{CH}_{\dim(\mathcal{X})-\bullet}(\mathcal{X}).$$

with cohomological grading, assuming the latter is defined (for example, if it has a stratification by quotient stacks). All algebraic stacks are assumed to be defined over the field k.

3.1. Chow groups for algebraic stacks admitting a good filtration.

In [Kre99], Chow groups, and the intersection ring for smooth algebraic stacks, are only defined for an algebraic stack of finite type over k. We will need to extend the definition to stacks which are close enough to being of finite type for the purpose of intersection theory.

Definition 5. A *good filtration by finite-type substacks* on an algebraic stack $\mathcal{X}$, which is assumed to be locally of finite type and of finite dimension, is a collection $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ such that

- $\mathcal{X}_n \subset \mathcal{X}$ is an open substack of finite type;
- $\mathcal{X}_n \subset \mathcal{X}_m$ for $n < m$;
- $\dim(\mathcal{X} \setminus \mathcal{X}_n) < \dim X - n$.

In this situation, we will abbreviate to say that $\mathcal{X}$ *admits a good filtration*. A morphism of algebraic stacks admitting a good filtration *respects the filtration* if it factors through the filtrations in the natural way, up to a degree shift.

Remark. The last condition could be weakened to only requiring that the $\mathcal{X}_n$ jointly cover $\mathcal{X}$, and that $\lim_{n\to\infty} \operatorname{codim}(\cup_{i=0}^n \mathcal{X}_i) = \infty$; however, all the stacks in our example already come with a natural good filtration, and it is possible to pass from any filtration by open substacks of finite type to a good filtration.

Proposition 6. *There is a Chow group functor* $\mathrm{CH}_d(\mathcal{X}) := \varprojlim_n \mathrm{CH}_d(\mathcal{X}_n)$ *defined for stacks admitting a good filtration, satisfying the usual properties as in* [Kre99], *with functoriality for morphisms of algebraic stacks which respect the filtration. In fact, we have* $\mathrm{CH}_d(\mathcal{X}) = \mathrm{CH}_d(\mathcal{X}_{\dim(\mathcal{X})+d})$. *If* $\mathcal{X}$ *is smooth and admits a stratification by quotients stacks, there is also a ring structure defined on the Chow groups, and we can compute the product* $\mathrm{CH}^{d'}(\mathcal{X}) \otimes \mathrm{CH}^{d''}(\mathcal{X}) \to \mathrm{CH}^{d'+d''}(\mathcal{X})$ *on* $\mathcal{X}_{d'+d''}$.

Proof. This is a straightforward consequence of excision. □

All of the algebraic stacks appearing in this note admit a good filtration, and we will use the colimit above implicitly when discussing their Chow groups.

3.2. **Étale-local models.** We collect some facts about the stack quotients $\mathcal{A}^n := [\mathbb{A}^n/\mathbb{G}_m^n] = (\mathcal{A}^1)^{\times n}$ for $n \in \mathbb{Z}_{\geq 0}$ and their Chow rings. Recall that $\mathcal{A}^1$ is the moduli stack of pairs $(\mathcal{L}, s)$ of a line bundle $\mathcal{L}$ and a section s of $\mathcal{L}$, and hence $\mathcal{A}^n$ is the moduli stack of n-tuples of such pairs.

Since $\mathcal{A}^n$ is a vector bundle of rank n over $B\mathbb{G}_m^n \cong (B\mathbb{G}_m)^{\times n}$, the pullback of cycle classes from $B\mathbb{G}_m^n$ is an isomorphism. Let $[\mathbb{A}^1/\mathbb{G}_m^n]_i$ correspond to the one-dimensional representation of $B\mathbb{G}_m^n$ given by $x \mapsto t_i x$.

Proposition 7. *The graded ring* $\mathrm{CH}^\bullet(B\mathbb{G}_m^n)$ *is isomorphic to* $\mathbb{Z}[\alpha_1, \ldots, \alpha_n]$, *where* α_i *corresponds to the class of the zero section of* $[\mathbb{A}^1/\mathbb{G}_m^n]_i$ *under the Gysin homomorphism* $\mathrm{CH}^\bullet([\mathbb{A}^1/\mathbb{G}_m^n]_i) \to \mathrm{CH}^\bullet(B\mathbb{G}_m^n)$.

Corollary 8. *We have*

$$\mathrm{CH}^\bullet(\mathcal{A}^n) \cong \mathbb{Z}[\alpha_1, \ldots, \alpha_n].$$

Lemma 9. *The normal bundle of* $B\mathbb{G}_m^n$ *in* $\mathcal{A}^n$ *has top Chern class* $\alpha_1 \cdots \alpha_n$.

Proof. We can compute the normal bundle on any smooth atlas. Choose the atlas $\mathbb{A}^n$ to find that the normal bundle is $\mathcal{A}^n$ itself, seen as a vector bundle over $B\mathbb{G}_m$. This a sum of line bundles

$$N_{B\mathbb{G}_m^n}\mathcal{A}^n = \bigoplus_{i=1}^n [\mathbb{A}^1/\mathbb{G}_m^n]_i.$$

Hence we have

$$c_n(N_{B\mathbb{G}_m^n}\mathcal{A}^n) = \prod_{i=1}^n \alpha_i.$$

□

Definition 6. We say that an algebraic stack $\mathfrak{Y}$ over k has the *Chow Künneth property* if for all algebraic stacks $\mathcal{X}$ of finite type over k, the natural morphism

$$\mathrm{CH}^\bullet(\mathcal{X}) \otimes \mathrm{CH}^\bullet(\mathfrak{Y}) \to \mathrm{CH}^\bullet(\mathcal{X} \times_k \mathfrak{Y}) \tag{4}$$

induced by functoriality is an isomorphism.

Remark. If $\mathfrak{Y}$ has the Chow Künneth property, then (4) will also be true if $\mathcal{X}$ only admits a good filtration.

Lemma 10. *The classifying stack $B\mathbb{G}_m$ has the Chow Künneth property.*

Proof. First note that for any $n \geq 1$, projective space $\mathbb{P}^{n-1}$ has the Chow Künneth property, by the formula for the Chow rings of projective bundles, specialized to the case of a trivial bundle (cf. [Ful98, 3.3]). To prove the lemma, note that $\mathcal{X} \times [\mathbb{A}^n/\mathbb{G}_m]$ is a vector bundle over $\mathcal{X} \times B\mathbb{G}_m$, which is $\mathcal{X} \times \mathbb{P}^{n-1}$ in codimension smaller than n.[4] By choosing n high enough and applying the Chow Künneth property of projective space, we obtain the statement in any fixed degree. □

As a consequence, also $\mathcal{A}^n$, for any n, has the Chow Künneth property. Occasionally, we will use coordinates indexed by a finite set J instead of $\{1, \dots, n\}$. In this situation, we will use the symbols $\mathbb{A}^J$ and $\mathcal{A}^J$.

3.3. **The stack of configurations of line bundles.** We continue to use the notation $\mathcal{A}^n$ from the last section. In this section, we introduce the stack of expanded pairs $\mathcal{T}$. The following treatment has been adapted from [ACFW13]. There are many equivalent moduli definitions (and universal families) for $\mathcal{T}$; we will describe one of them for the convenience of the reader, where $\mathcal{T}$ appears in the form of the moduli stack of *configurations of line bundles*. This point of view has been inspired from the approach to logarithmic structures in terms of line bundles in [BV12].

Definition 7. A *sheaf of totally ordered finite sets* is a constructible sheaf E of partially ordered nonempty[5] finite sets on the étale site of a scheme S such that any two sections are locally comparable. We identify E with the stack on the étale site of S whose objects are sections of E, and where a unique morphism $x \to y$ exists if and only if $x \geq y$.

We let $\mathfrak{Pic}$ be the category of line bundles. The objects of $\mathfrak{Pic}$ over S are line bundles on S, and its morphisms are morphisms of line bundles. We should remark that morphisms are *not* required to be isomorphisms of line bundles, so $\mathfrak{Pic}$ is not a category fibered in groupoids, as opposed to $B\mathbb{G}_m$.

Definition 8. Let $\mathcal{T}$ be the stack whose objects are pairs (E, L) of a sheaf of totally ordered finite sets E and a morphism of stacks $L : E \to \mathfrak{Pic}$ such that

[4] In the topological setting, we have $BU(1) = \mathbb{CP}^\infty$. In this spirit, one could regard $B\mathbb{G}_m$ as an algebraic version of $\mathbb{P}^\infty$.

[5] except over the empty set.

(1) If $x \geq y$ are sections of E and $L(x) \to L(y)$ is an isomorphism, then $x = y$.
(2) $L(0) = \mathcal{O}$, where 0 is the unique section of E that is minimal in all fibers.

We call such an L a *diagram of line bundles* indexed by E. We require that morphisms in $\mathfrak{T}$ are the identity on $L(0)$. We call $\mathfrak{T}$ the stack of totally ordered configurations of line bundles.

We recall the definition of the moduli stack of genus g semistable curves with n marked points $\mathfrak{M}_{g,n}^{ss} \subset \mathfrak{M}_{g,n}$, whose objects are prestable curves of genus g endowed with n sections such that the relative dualizing sheaf, twisted by the sections, has non-negative multidegree. It is classical that $\mathfrak{M}_{g,n}^{ss}$ is an algebraic stack locally of finite presentation over k.

We will only need the genus 0 case; for $n > 0$, its geometric points are nodal curves such that each component is isomorphic to projective space and such that each component has at least two special points (either nodes or marked points).

Theorem 2 ([ACFW13])**.** *There is an isomorphism of $\mathfrak{T}$ with the open substack of $\mathfrak{M}_{0,3}^{ss}$ where the last two points lie on the same component.*

There is also a useful description of $\mathfrak{T}$ as a colimit. Given a finite, possibly empty, ordered set J, there is a natural augmentation $\tilde{J}$, which is the union $J \sqcup \{0\}$ of I and a smallest element 0. An order-preserving injection of finite sets $J \to K$ induces an open embedding $\mathcal{A}^J \to \mathcal{A}^K$. Assigning $J \mapsto \mathcal{A}^J$, together with the above morphisms, give rise to diagrams of algebraic stacks indexed by $\mathcal{C}_{\leq k}$, respectively $\mathcal{C}$.

There are natural diagrams of line bundles on $\mathcal{A}^J$ as follows. Consider the universal family on $\mathcal{A}^J$, that is, a collection $((\mathcal{L}_i, s_i))_{i \in J}$ of line bundles and sections. Let E be the quotient of the constant sheaf $\tilde{J}$ on $\mathcal{A}^J$ by the relation $i \sim i+1$ on the locus where s_{i+1} is nonzero. Denote the elements of J by $\{1 < \cdots < n\}$ for $|J| = n$. Then the following sequence of morphisms of line bundles defines a morphism $\tilde{J} \to \mathfrak{Pic}$ which descends to a diagram of line bundles on $\mathcal{A}^J$ indexed by E.

$$L_1^\vee \otimes \cdots \otimes L_n^\vee \xrightarrow{s_n} L_1^\vee \otimes \cdots \otimes L_{n-1}^\vee \xrightarrow{s_{n-1}} \ldots \xrightarrow{s_2} L_1^\vee \xrightarrow{s_1} \mathcal{O}$$

This gives rise to morphisms $\mathcal{A}^J \to \mathfrak{T}$ for every J, compatible with the embeddings $\mathcal{A}^J \hookrightarrow \mathcal{A}^K$.

Proposition 11 ([ACFW13, Prop 8.3.1])**.** *These morphisms are étale and induce an equivalence*

$$\varinjlim_{\mathcal{C}} \mathcal{A}^J \xrightarrow{\sim} \mathfrak{T}. \tag{5}$$

Setting

$$\mathfrak{T}^{\leq d} := \varinjlim_{\mathcal{C}_{\leq d}} \mathcal{A}^J \hookrightarrow \mathfrak{T},$$

we see that the collection $\{\mathfrak{T}^{\leq d}\}_k$ forms a good filtration of $\mathfrak{T}$. The following observation will be useful: $\mathfrak{T}^{\leq d}$ possesses a unique closed point with stabilizer $B\mathbb{G}_m^d$, whose complement is $\mathfrak{T}^{\leq d-1}$. It pulls back to the image of the origin in $\mathcal{A}^J$ for $|J| = d$. We will sometimes call a geometric point of $\mathfrak{T}$ over this point an *accordion of length d*, according to the geometric picture in $\mathfrak{M}_{0,3}^{ss}$: it is a nodal curve which is a chain of $d+1$ projective spaces with one marking on one end two markings on the other end, and the action of the automorphism group $\mathbb{G}_m^d$ looks like an accordion being played.

4. Cycle group calculations

4.1. **The Chow ring of $\mathfrak{T}$.** We will prove the following result.

Theorem 3. *Let $\mathcal{X}$ be an algebraic stack, smooth over k and admitting a good filtration, which has a stratification by quotient stacks.*

(1) The natural morphism

$$\mathrm{CH}^\bullet(\mathcal{X} \times \mathfrak{T}^{\leq d}) = \mathrm{CH}^\bullet(\varinjlim_{\mathcal{C}_{\leq d}} \mathcal{X} \times \mathcal{A}^n) \to \varprojlim_{\mathcal{C}_{\leq d}} \mathrm{CH}^\bullet(\mathcal{X} \times \mathcal{A}^n) = \mathrm{CH}^\bullet(\mathcal{X}) \otimes \mathrm{QSym}_d \quad (6)$$

induced (via functoriality with regard to étale morphisms) by the colimit (5) *is an isomorphism for all d.*

(2) The natural morphism

$$\mathrm{CH}^\bullet(\mathcal{X} \times \mathfrak{T}) = \mathrm{CH}^\bullet(\varinjlim_{\mathcal{C}} \mathcal{X} \times \mathcal{A}^n) \to \varprojlim_{\mathcal{C}} \mathrm{CH}^\bullet(\mathcal{X} \times \mathcal{A}^n) = \mathrm{CH}^\bullet(\mathcal{X}) \otimes \mathrm{QSym} \quad (7)$$

is an isomorphism.

In particular, the stack $\mathfrak{T}$ has the Chow Künneth property, and $\mathrm{CH}^\bullet(\mathfrak{T}) \cong \mathrm{QSym}$.

Remark 1. If $\mathcal{X}$ admits a good filtration, but is not smooth, Theorem 3 is still true, using almost the same proof, for the Chow groups without ring structure.

Proof. First we note that the second part follows from the first. To see this, note that the complement of $\mathfrak{T}^{\leq d}$ has codimension $d+1$, hence for any $d' \leq d$,

$$\mathrm{CH}^{d'}(\mathfrak{T} \times \mathcal{X}) \cong \mathrm{CH}^{d'}(\mathfrak{T}^{\leq d} \times \mathcal{X})$$

by excision. To prove the first part, we fix notation by setting $B := \mathrm{CH}^\bullet(\mathcal{X})$. Also, for any algebraic stack $\mathfrak{Y}$, let $\mathfrak{Y}_{\mathcal{X}} := \mathcal{X} \times \mathfrak{Y}$, and for morphisms $f : \mathfrak{Y} \to \mathfrak{Z}$, let $f_{\mathcal{X}} := \mathrm{id} \times f : \mathfrak{Y}_{\mathcal{X}} \to \mathfrak{Z}_{\mathcal{X}}$.

We prove the statement by induction on d. More precisely, we prove by induction that the morphism $\mathrm{CH}^\bullet(\mathfrak{T}^{\leq d}_{\mathcal{X}}) \to \mathrm{CH}^\bullet(\mathcal{A}^d_{\mathcal{X}})$ is injective, with image equal to $B \otimes \mathrm{QSym}_d$. The case $d = 0$ is obvious. Suppose the theorem has been proven for $d-1$. By Lemma 9, the map $\mathrm{CH}^\bullet((B\mathbb{G}_m^d)_{\mathcal{X}}) \to \mathrm{CH}^{\bullet+d}(\mathcal{A}^d_{\mathcal{X}})$ induced by inclusion is injective. Since $(p_d)_{\mathcal{X}}$ is étale, the same holds for $\mathrm{CH}^\bullet((B\mathbb{G}_m^d)_{\mathcal{X}}) \to \mathrm{CH}^{\bullet+d}(\mathfrak{T}^{\leq d}_{\mathcal{X}})$. We will determine its image. By excision, the following commutative diagram is exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{CH}^\bullet((B\mathbb{G}_m^d)_{\mathcal{X}}) & \longrightarrow & \mathrm{CH}^{\bullet+d}(\mathcal{A}^d_{\mathcal{X}}) & \longrightarrow & \mathrm{CH}^{\bullet+d}((\mathcal{A}^d \setminus B\mathbb{G}_m^d)_{\mathcal{X}}) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle (id_{B\mathbb{G}_m^d})^*_{\mathcal{X}}} & & \uparrow{\scriptstyle (p_d)^*_{\mathcal{X}}} & & \uparrow{\scriptstyle (p_d|_{(\mathcal{A}^d \setminus B\mathbb{G}_m^d)})^*_{\mathcal{X}}} & & \\ 0 & \longrightarrow & \mathrm{CH}^\bullet((B\mathbb{G}_m^d)_{\mathcal{X}}) & \longrightarrow & \mathrm{CH}^{\bullet+d}(\mathfrak{T}^{\leq d}_{\mathcal{X}}) & \longrightarrow & \mathrm{CH}^{\bullet+d}(\mathfrak{T}^{\leq d-1}_{\mathcal{X}}) & \longrightarrow & 0 \end{array} \quad (8)$$

Let $\psi_i : \mathcal{A}_{\mathcal{X}}^{d-1} \to \mathcal{A}_{\mathcal{X}}^{d}$ induced by j_i^{d-1}. Consider the following (not necessarily commutative) diagram.

$$\begin{tikzcd}
\mathrm{CH}^\bullet((\mathcal{A}^d \setminus B\mathbb{G}_m^d)_{\mathcal{X}}) \arrow[dr, "\psi_i^*"'] \arrow[drr, "\psi_j^*"] & & \\
& \mathrm{CH}^\bullet(\mathcal{A}_{\mathcal{X}}^{d-1}) \arrow[r, "\sim"] & \mathrm{CH}^\bullet(\mathcal{A}_{\mathcal{X}}^{d-1}) \\
\mathrm{CH}^\bullet(\mathcal{T}_{\mathcal{X}}^{\leq d-1}) \arrow[uu, "(p_d|_{(\mathcal{A}^d\setminus B\mathbb{G}_m^d)})^*_{\mathcal{X}}"] \arrow[ur, "(p_{d-1})^*_{\mathcal{X}}"] \arrow[urr, "(p_{d-1})^*_{\mathcal{X}}"'] & &
\end{tikzcd} \tag{9}$$

We take the horizontal map h to be induced by the order-preserving bijection $\{1, \dots, \hat{j}, \dots, d\} \to \{1, \dots, \hat{\imath}, \dots, d\}$. Then, because $\mathcal{T}_{\mathcal{X}}^{d-1}$ is a colimit over $\mathcal{C}_{\leq d-1}$,

$$h \circ \psi_i^* \circ (p_d|_{(\mathcal{A}^d\setminus B\mathbb{G}_m^d)})^*_{\mathcal{X}} = (p_{d-1})^*_{\mathcal{X}} = \psi_j^* \circ (p_d|_{(\mathcal{A}^d\setminus B\mathbb{G}_m^d)})^*_{\mathcal{X}}. \tag{10}$$

Since $(p_{d-1})^*_{\mathcal{X}}$ is injective by induction hypothesis, so is $(p_d|_{(\mathcal{A}^d\setminus B\mathbb{G}_m^d)})^*_{\mathcal{X}}$. By the five lemma, applied to (8), we deduce that $(p_d)^*_{\mathcal{X}}$ is injective. First, fill in the known (by the induction hypothesis) terms in (8).

$$\begin{tikzcd}
0 \arrow[r] & B \otimes \mathbb{Z}[\alpha_i]_{i=1}^d \arrow[r, "\cdot\alpha_1\cdots\alpha_d"] & B \otimes \mathbb{Z}[\alpha_i]_{i=1}^d \arrow[r] & B \otimes \mathbb{Z}[\alpha_i]_{i=1}^d/(\alpha_1 \cdots \alpha_d) \arrow[r] & 0 \\
0 \arrow[r] & B \otimes \mathbb{Z}[\alpha_i]_{i=1}^d \arrow[r] \arrow[u, "id"] & \mathrm{CH}^\bullet(\mathcal{T}_{\mathcal{X}}^{\leq d}) \arrow[r] \arrow[u, "(p_d)^*_{\mathcal{X}}"] & B \otimes \mathrm{QSym}_{d-1} \arrow[r] \arrow[u, "(p_d|_{(\mathcal{A}^d\setminus B\mathbb{G}_m^d)})^*_{\mathcal{X}}"] & 0
\end{tikzcd} \tag{11}$$

Likewise, we fill in the known groups in (9).

$$\begin{tikzcd}
B \otimes \mathbb{Z}[\alpha_i]_{i=1}^d/(\alpha_1 \cdots \alpha_d) \arrow[dr, "\alpha_i \mapsto 0"'] \arrow[drr, "\alpha_j \mapsto 0"] & & \\
& B \otimes \mathbb{Z}[\alpha_\ell]_{\ell=1,\ell\neq i}^n \arrow[r, "\sim"] & B \otimes \mathbb{Z}[\alpha_\ell]_{\ell=1,\ell\neq j}^n \\
B \otimes \mathrm{QSym}_{d-1} \arrow[uu, "(p_d|_{(\mathcal{A}^d\setminus B\mathbb{G}_m^d)})^*_{\mathcal{X}}"] \arrow[ur, "(p_{d-1})^*_{\mathcal{X}}"] \arrow[urr, "(p_{d-1})^*_{\mathcal{X}}"'] & &
\end{tikzcd} \tag{12}$$

To conclude, we apply the induction hypothesis and Lemma 12. The latter implies that the image of $\mathrm{CH}^\bullet(\mathcal{T}_{\mathcal{X}}^{\leq d})$ in $B \otimes \mathbb{Z}[\alpha_1, \dots, \alpha_d]$ under $(p_d)^*_{\mathcal{X}}$ can be identified with

$$B \otimes \left(\mathbb{Z}[\alpha_1, \dots, \alpha_d] \times_{\mathbb{Z}[\alpha_1,\dots,\alpha_d]/(\alpha_1\cdots\alpha_d)} \mathrm{QSym}_{d-1}\right).$$

By the induction hypothesis, the morphism $(p_{d-1})^*_{\mathcal{X}}$ is the inclusion of $B \otimes \mathrm{QSym}_{d-1}$ into a polynomial ring over B with $d-1$ ordered variables. Hence, by (10), the image of $B \otimes \mathrm{QSym}_{d-1}$ in $B \otimes \mathbb{Z}[\alpha_i]_{i=1}^d/(\alpha_1 \cdots \alpha_d)$ is exactly equal to (the image in $B \otimes \mathbb{Z}[\alpha_i]_{i=1}^d/(\alpha_1 \cdots \alpha_d)$ of) those $f \in B \otimes \mathrm{QSym}_d$ such that $f \notin (\alpha_1 \cdots \alpha_d)$. By the commutativity of (8), this implies that the image of $(p_d)^*_{\mathcal{X}}$ contains a representative mod $(\alpha_1 \cdots \alpha_d)$ of every $f \in B \otimes \mathrm{QSym}_d$ such that $f \notin (\alpha_1 \cdots \alpha_d)$, or to say it differently, it contains a representative mod $(\alpha_1 \cdots \alpha_d)$ of every quasi-symmetric polynomial with B-coefficients that has weight less than d. Furthermore, it also contains $(\alpha_1, \dots, \alpha_d) \subset \mathrm{QSym}_d$ (from the left-hand side of the fiber product). Via a direct computation using the colimit description and functoriality, it is immediate that any polynomial in the image must be quasisymmetric, and we have shown that the image contains all quasisymmetric polynomials. This concludes the proof. □

Lemma 12 ([VV03, Lemma 4.4], graded variant)**.** *Let A, B, and C be graded rings, and let $f : B \to A$ and let $g : B \to C$ be morphisms. Suppose that there exists a homomorphism of abelian groups $\phi : A \to B$ such that:*

(1) The sequence

$$0 \to A \xrightarrow{\phi} B \xrightarrow{g} C \to 0$$

is exact;

(2) the composition $f \circ \phi : A \to A$ is the multiplication by an element $a \in A$ of pure degree which is not a zero-divisor.

Then f and g induce an isomorphism of graded rings

$$(f, g) : B \to A \times_{A/(a)} C,$$

where $A \to A/(a)$ is the projection and $C \to A/(a)$ is induced by $C \cong B/\operatorname{im}(\phi) \to A/\operatorname{im}(f \circ \phi) = A/(a)$. □

Remark 2. The geometric meaning of most of the classes in $\mathrm{CH}^\bullet(\mathcal{T})$ is somewhat mysterious to the author. The subring of *symmetric* functions is the ring generated by closed substacks: it follows from the previous calculation that the class of $\mathcal{T}^{\geq d}$ is M_I, where

$$I = \underbrace{(1, \dots, 1)}_{d \text{ times}}.$$

In other words, it is the d-th elementary symmetric function. On the other hand, consider a generator L of the Picard group of $\mathcal{A}^1$. The unique (up to isomorphism) line bundle on $\mathcal{T}$ whose restriction to the open substack $\mathcal{A}^1 \subset \mathcal{T}$ is isomorphic to L has first Chern class $\pm M_{(1)}$. Hence, we readily obtain powers of $M_{(1)}$ as Chern classes of vector bundles, for example

$$M_{(1)}^2 = M_{(2)} + 2M_{(1,1)}.$$

However, it is not clear how we can naturally produce non-symmetric classes such as $M_{(1,2)}$.

4.2. **Hopf algebra structure.** We construct a morphism $\mu : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ that gives rise to the comultiplication of the Hopf algebra. Given two pairs (E_1, L_1) of (E_2, L_2) of a sheaf of totally ordered sets and a diagram of line bundles indexed by that sheaf, we define E as the sheaf of partially ordered sets that arises as the sheafification of the presheaf that identifies the unique minimal element of E_2 with the highest element of E_1. Then

$$L : E \to \mathfrak{Pic}$$

is the morphism which takes a section x to $L_1(x)$ if x is a section of E_1, and to $L_2(x) \otimes \tilde{L}$ if x is a section of E_2, where $\tilde{L}$ is the image under L_1 of the local maximum of E_1.

Informally, we associate to two sequences of line bundles and morphisms

$$L_{1,n_1} \to \cdots \to L_{1,1} \to \mathcal{O} \text{ and } L_{2,n_2} \to \cdots \to L_{2,1} \to \mathcal{O}$$

the sequence

$$\begin{aligned} L_{2,n_2} \otimes L_{1,n_1} \to L_{2,n_2-1} \otimes L_{1,n_1} \to \cdots \to L_{2,1} \otimes L_{1,n_1} \\ \to L_{1,n_1} \to L_{1,n_1-1} \to \cdots \to L_{1,1} \to \mathcal{O}. \end{aligned}$$

For $n = n_1 + n_2$, we can identify $\Psi_{n_1,n_2} : \mathcal{A}^{n_1} \times \mathcal{A}^{n_2} \xrightarrow{\simeq} \mathcal{A}^n$ under the obvious isomorphism preserving the order of the coordinates. Since the former form an étale cover of $\mathcal{T} \times \mathcal{T}$ and the latter form an étale cover of $\mathcal{T}$, it will be important so understand how they interact.

Proposition 13. *Let $n, n_1, n_2, \Psi_{n_1,n_2}$ as above. There is an isomorphism*

$$\mu \circ (p_{n_1} \times p_{n_2}) \simeq p_n \circ \Psi_{n_1,n_2}.$$

Proof. This is clear by the definition of μ. □

Lemma 14. *The morphism μ defined by the previous construction is representable étale.*

Proof. Since the stabilizer groups of an accordion of length d is $\mathbb{G}_m^d$ and the morphism maps a pair of accordions of length d_1 and d_2 to an accordion of length d_1+d_2, the morphism is stabilizer-preserving, hence representable. For representable morphisms, we can check the property of being étale on an atlas for the target. Since the property of being étale is étale local on the source, it is enough to show that

$$\mathcal{A}^{n_1} \times \mathcal{A}^{n_2} \to \mathcal{T}^{\leq n}$$

is étale whenever $n_1 + n_2 \leq n$, and for that it is enough to show that it is étale in the case $n_1 + n_2 = n$. After base change by $\mathcal{A}^n \to \mathcal{T}^{\leq n}$, we have to show that the 2-fiber product

$$(\mathcal{A}^{n_1} \times \mathcal{A}^{n_2}) \times_{\mathcal{T}} \mathcal{A}^n$$

is étale over $\mathcal{A}^n$, but this is just the diagonal of $\mathcal{A}^n \to \mathcal{T}$. □

Remark 3. One should note that while μ is étale and quasi-finite, it is not separated, since it admits sections. For example, we have the embeddings $\mathcal{T} \cong \mathcal{T} \times \mathcal{T}^{\leq 0} \hookrightarrow \mathcal{T} \times \mathcal{T}$ and $\mathcal{T} \cong \mathcal{T}^{\leq 0} \times \mathcal{T} \hookrightarrow \mathcal{T} \times \mathcal{T}$, both of which are sections of μ.

Remark 4. As suggested by the notation, the morphism μ is the multiplication morphism exhibiting $(\mathcal{T}, \mu, \mathrm{Spec}(k) \hookrightarrow \mathcal{T})$ as a monoid object in the 2-category of algebraic stacks.

Theorem 4. *The comultiplication $\Delta : \mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym}$ is equal to the pullback of cycle classes μ^*.*

Proof. It is enough to show that $\mu^*(M_I) = \Delta(M_I)$ for every composition I, where M_I is the monomial basis element indexed by I. Fix I and let $n = |I|$ be the size of I, which is also the degree of M_I. We have established in the proof of Theorem 3 that the pullback of cycle classes on $\mathcal{T}$ of cohomological degree at most n to $\mathcal{A}^n$ is injective. The Chow ring of $\mathcal{T} \times \mathcal{T}$ is naturally bigraded as a tensor product of graded algebras. Using the same technique twice, we can see that for cycle classes of bidegree at most (n_1, n_2), the pullback to $\mathcal{A}^{n_1} \times \mathcal{A}^{n_2}$ is injective.

Note that the complement of the union of the images of $\mathcal{A}^{n_1} \times \mathcal{A}^{n_2}$ as we range over pairs (n_1, n_2) with $n_1 + n_2 = n$ has codimension $n + 1$. This implies that for any $d \leq n$, a class in $\mathrm{CH}^d(\mathcal{T} \times \mathcal{T})$ can be uniquely identified by its image in

$$\prod_{n_1+n_2=n} \mathrm{CH}^d(\mathcal{A}^{n_1} \times \mathcal{A}^{n_2})$$

by the product of the various pullbacks, by excision([Kre99, Thm. 2.1.12(iv-v)]). It remains to be shown that the pullback of $\mu^* M_I$ to $\mathrm{CH}^n(\mathcal{A}^{n_1} \times \mathcal{A}^{n_2})$ is equal to the pullback of

$$\sum_{s=0}^{\ell} M_{(I_1,\dots,I_s)} \otimes M_{(I_{s+1},\dots,I_\ell)}. \tag{13}$$

Let $\Psi := \Psi_{n_1,n_2}$. We use the fact that $\mu \circ (p_{n_1} \times p_{n_2})$ is isomorphic to $p_n \circ \Psi$.

Under this isomorphism, the monomial M_I corresponding to a composition $I = (I_1, \dots, I_\ell)$ pulls back to the same monomial in QSym $\otimes$ QSym, except that we regard the first n_1 variables as coming from the left-hand side of the tensor product, and the remaining n_2 variables as coming from the right-hand side.

To make this precise, let us denote the variables in $\mathrm{CH}^\bullet(\mathcal{A}^n), \mathrm{CH}^\bullet(\mathcal{A}^{n_1})$, and $\mathrm{CH}^\bullet(\mathcal{A}^{n_2})$ by γ_i, α_i, and β_i respectively, such that $\Psi^*\gamma_i = \alpha_i$ for $i \in \{1, \dots, n_1\}$ and $\Psi^*\gamma_{n_1+i} = \beta_i$ for $i \in \{1, \dots, n_2\}$. Then we see that

$$\begin{aligned}
\Psi^* M_I &= \Psi^* \left(\sum_{i_1 < \cdots < i_\ell} \gamma_{i_1}^{I_1} \cdots \gamma_{i_\ell}^{I_\ell} \right) \\
&= \sum_{i_1 < \cdots < i_\ell} \Psi^*(\gamma_{i_1})^{I_1} \cdots \Psi^*(\gamma_{i_\ell})^{I_\ell} \\
&= \sum_{\substack{i_1 < \cdots < i_k \leq n_1 \\ n_1 < i_{k+1} < \cdots < i_\ell}} \Psi^*(\gamma_{i_1})^{I_1} \cdots \Psi^*(\gamma_{i_k})^{I_k} \Psi^*(\gamma_{i_{k+1}})^{I_{k+1}} \cdots \Psi^*(\gamma_{i_\ell})^{I_\ell} \\
&= \sum_{\substack{i_1 < \cdots < i_k \leq n_1 \\ n_1 < i_{k+1} < \cdots < i_\ell}} (\alpha_{i_1} \otimes 1)^{I_1} \cdots (\alpha_{i_k} \otimes 1)^{I_k} (1 \otimes \beta_{(i_{k+1}-n_1)})^{I_{k+1}} \cdots (1 \otimes \beta_{(i_\ell - n_1)})^{I_\ell} \\
&= \sum_{\substack{I = J \cdot K \\ J \in \mathrm{Comp}_{n_1} \\ K \in \mathrm{Comp}_{n_2}}} M_J \otimes M_K = \sum_{\substack{I = J \cdot K \\ \ell(J) \leq n_1 \\ \ell(K) \leq n_2}} M_J \otimes M_K.
\end{aligned} \tag{14}$$

On the other hand, the pullback of $p(\alpha_i) \otimes q(\beta_i) \in$ QSym $\otimes$ QSym by $(p_{n_1} \times p_{n_2})$ is the evaluation homomorphism that sets all α_i to 0 for $i > n_1$ and all β_i to 0 for $i > n_2$, and is the identity on the remaining variables. Hence

$$(p_{n_1} \times p_{n_2})^* M_J \otimes M_K = \begin{cases} M_J \otimes M_K, & \text{if } \ell(J) \leq n_1 \text{ and } \ell(K) \leq n_2 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, by applying the previous equation to (13), we see that

$$(p_{n_1} \times p_{n_2})^*(\Delta(M_I))$$

is equal to the sum (14). $\square$

Remark. Since the structure morphism $\mathfrak{T} \to \operatorname{Spec} k$ and the inclusion of the generic point $\operatorname{Spec} k \to \mathfrak{T}$ induce the unit and counit of QSym, respectively, one could conjecture that there could also be a morphism $\widehat{S} : \mathfrak{T} \to \mathfrak{T}$ inducing the antipode. However, since the antipode takes the class $M_{(1)}$ of the unique effective primitive divisor of $\mathfrak{T}$ to $-M_{(1)}$, a geometric antipode does not exist.

4.3. **A natural involution.** The stack $\mathfrak{T}$ comes equipped with an involution σ, which sends a diagram of line bundles

$$L_n \to L_{n-1} \to \cdots \to L_1 \to \mathcal{O}$$

to the "dualized" diagram

$$L_n \xrightarrow{s_1^\vee \otimes \mathrm{id}} L_1^\vee \otimes L_n \to \ldots \to L_{n-1}^\vee \otimes L_n \xrightarrow{s_n^\vee \otimes \mathrm{id}} L_n^\vee \otimes L_n \cong \mathcal{O}.$$

This corresponds to the involution $\operatorname{rev}\colon \mathfrak{C} \to \mathfrak{C}$ of the index categeory $\mathfrak{C}$ which sends a finite ordered set to the same set with the opposite order, and similarly for morphisms. There is an explicit description of σ^* as follows.

Proposition 15. *The homomorphism of graded algebras* $\sigma^* : \mathrm{QSym} \to \mathrm{QSym}$ *is characterized in the monomial basis by*

$$\sigma^* M_J = M_{\operatorname{rev}(J)}.$$

This is the unique nontrivial graded algebra automorphism of QSym *which preserves the monomial basis.*

Proof. It is enough to show that the restriction $\sigma|_{\mathfrak{T}^{\leq d}}$ induces the claimed automorphism for each d. Let $J = [d]$, and consider the étale covers $p\colon \mathcal{A}^J \to \mathfrak{T}^{\leq d}$ and $\tilde{p}\colon \mathcal{A}^{\operatorname{rev}(J)} \to \mathfrak{T}^{\leq d}$. The involution $\sigma|_{\mathfrak{T}^{\leq d}}$ does not lift to a morphism $\mathcal{A}^J \to \mathcal{A}^J$ over $\mathfrak{T}^{\leq d}$, but it lifts to the identity $\mathrm{id}\colon \mathcal{A}^J \to \mathcal{A}^{\operatorname{rev}(J)}$, which is not induced by any morphism in $\mathfrak{C}$. Let $J = (J_1, \ldots, J_\ell)$ be a composition of length at most d. The monomial $M_J \in \mathrm{CH}(\mathfrak{T}^{\leq d})$ pulls back to

$$\sum_{i_1 < \cdots < i_\ell} \alpha_{i_1}^{J_1} \cdots \alpha_{i_\ell}^{J_\ell}$$

in $\mathrm{CH}(\mathcal{A}^J)$ respectively $\mathrm{CH}(\mathcal{A}^{\operatorname{rev}(J)})$, where the indices i_j are taken from the same set $[n]$, but the order relation is inverted. Hence we see that $\tilde{p}^* \sigma^* M_J = p^* M_{\operatorname{rev}(J)}$. For the second part, see [JWY17]. $\square$

Note that there does not exist any nontrivial algebra automorphism of QSym which preserves the monomial basis *and* respects the comultiplication.

5. Application to $\mathfrak{M}_{0,2}^{ss}$ and $\mathfrak{M}_{0,3}^{ss}$

We can now leverage our calculations and the interpretation of $\mathfrak{T}$ as a substack of $\mathfrak{M}_{0,3}^{ss}$ to obtain some immediate corollaries for the Chow groups of $\mathfrak{M}_{0,n}^{ss}$ for $n \in \{2, 3\}$.

Consider the stack $\mathfrak{M}_{0,3}^{ss}$ of semistable curves of genus 0 with 3 marked points. Note that any such curve over k has precisely one stable component. We can define a morphism $\Psi : \mathfrak{T} \times \mathfrak{T} \times \mathfrak{T} \to \mathfrak{M}_{0,3}^{ss}$ as follows: Given three curves C_1, C_2, C_3 with markings $(\sigma_1, \ldots, \sigma_9)$ such that 2 markings on each curve lie on the same component, glue the stable components along the three special points (in such a way that one marked points in C_1 is glued to the node in C_2, and the other marked points to the node in C_3, and similarly for marked points on the other curves). The result is a nodal semistable curve with 3 marked points and precisely one stable component. A triple of morphisms in $\mathcal{T}$, i.e. Cartesian diagrams, is taken to the glued Cartesian diagram. This is an isomorphism, because it is a representable étale morphism whose geometric fibers consist of single points. Theorem 3 then implies the following.

Corollary 16. *We have* $\mathrm{CH}^\bullet(\mathfrak{M}_{0,3}^{ss}) \cong \mathrm{QSym} \otimes \mathrm{QSym} \otimes \mathrm{QSym}$.

There is a variant of the stack of expansions, called the stack of non-rigid expansions, which can be shown to be isomorphic to $\mathfrak{M}_{0,2}^{ss}$. The two variants are connected by a rather simple relationship: moving one of the marked points on the stable component of $\mathcal{T}$, viewed as a stack of semistable curves, defines a $\mathbb{G}_m$-action, and $\mathfrak{M}_{0,2}^{ss}$ is the quotient by this action. More precisely, there is the following result:

Proposition 17 ([ACFW13, Prop. 3.3.4])**.** *There is a canonical (up to a twist by inversion) isomorphism* $\mathfrak{M}_{0,2}^{ss} \cong B\mathbb{G}_m \times \mathcal{T}$*, exhibiting* $\mathcal{T}$ *as the rigidification of* $\mathfrak{M}_{0,2}^{ss}$ *by the normal subgroup* $\mathbb{G}_m$ *of its inertia stack.*

Corollary 18. *We have* $\mathrm{CH}^\bullet(\mathfrak{M}_{0,2}^{ss}) \cong \mathrm{QSym}[\beta]$*, where* β *is the pullback of a generator under the morphism* $\mathfrak{M}_{0,2}^{ss} \to B\mathbb{G}_m$ *corresponding to the* $\mathbb{G}_m$*-action on* $\mathcal{T}$*.*

In fact, we can find a very explicit description of the isomorphism in Proposition 17 as follows: the stack $\mathfrak{M}_{0,2}^{ss}$ is isomorphic to the stack $\mathcal{T}^{\mathrm{nr}}$, whose definition is the same as Definition 8, except that we do not require condition (2).[6]

Then the isomorphism $\Phi : \mathcal{T}^{\mathrm{nr}} \to B\mathbb{G}_m \times \mathcal{T}$ takes a diagram

$$L_n \to \cdots \to L_1 \to L_0$$

to the pair

$$(L_0, L_n \otimes L_0^\vee \to \cdots \to L_1 \otimes L_0^\vee \to \mathcal{O}).$$

There is a natural involution $\tau : \mathfrak{M}_{0,2}^{ss} \to \mathfrak{M}_{0,2}^{ss}$ given by exchanging the two marked points. In the interpretation $\mathfrak{M}_{0,2}^{ss} \cong \mathcal{T}^{\mathrm{nr}}$, this corresponds to dualizing:

$$(L_n \to \cdots \to L_0) \mapsto (L_0^\vee \to \cdots \to L_n^\vee).$$

Proposition 19. *The action of* τ *on the Chow ring* $\mathrm{QSym}[\beta]$ *is given by mapping a quasisymmetric function* g *to* $\sigma^*(g)$ *for* σ *as in 4.3, and by mapping*

$$\beta \mapsto -\beta + M_{(1)} = -\beta + \sum_i \alpha_i.$$

Proof. First observe that τ restricts to σ on the second factor, so we only need to compute $\tau^*\beta$. Since τ^* is an automorphism of graded rings, it preserves the grading of τ, so it is enough to compute it on $B\mathbb{G}_m \times \mathcal{T}^{\leq 1} \cong B\mathbb{G}_m \times \mathcal{A}^1$, where we can see that the dual diagram $(L_0^\vee \to L_1^\vee)$ is sent by Φ to the pair

$$(L_1^\vee, L_0^\vee \otimes L_1 \to \mathcal{O}) = (L_0^\vee \otimes (L_0^\vee \otimes L_1)^\vee, L_0^\vee \otimes L_1 \to \mathcal{O}).$$

We conclude by remarking that $c_1((L_0^\vee \otimes L_1)^\vee) = M_{(1)}$. ☐

[6]The symbol “nr” stands for “non-rigid”.

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